

An L^3 bound for the dyadic triangular Hilbert form

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Abstract

We prove a uniform $L^3 \times L^3 \times L^3$ estimate for the dyadic triangular Hilbert form with unrestricted real inputs. The sum of the absolute local contributions over any finite set of scales is bounded by forty times the product of the input norms. In particular, the bound allows coefficients of modulus at most one to vary independently among admissible interval triples.

1 Introduction

The triangular Hilbert form couples three functions of two variables, each pair sharing one variable. We study its dyadic model, in which the intervals are constrained by binary addition without carries. The estimate below imposes no structural condition on any of the three inputs.

Write $\mathbb{R}_+ = [0, \infty)$. For $k \in \mathbb{Z}$, let

$$\mathcal{D}_k = \{[2^k n, 2^k(n+1)) : n \in \mathbb{Z}_{\geq 0}\}.$$

For $I \in \mathcal{D}_k$, denote its index by n_I and set

$$h_I = \mathbf{1}_{I_{\text{left}}} - \mathbf{1}_{I_{\text{right}}}.$$

Let $\oplus$ denote bitwise XOR of nonnegative integers, and define the set of admissible triples

$$\mathcal{A}_k = \{(I_0, I_1, I_2) \in \mathcal{D}_k^3 : n_{I_0} \oplus n_{I_1} \oplus n_{I_2} = 0\}.$$

For $\mathbf{I} \in \mathcal{A}_k$ and three real functions F_v on $\mathbb{R}_+^2$, put

$$L_{\mathbf{I}}(F_0, F_1, F_2) = 2^{-k} \int_{I_0 \times I_1 \times I_2} F_0(x_0, x_1) F_1(x_1, x_2) F_2(x_2, x_0) \prod_{v=0}^2 h_{I_v}(x_v) dx_0 dx_1 dx_2. \quad (1.1)$$

The dyadic triangular Hilbert form with a finite set of scales $S \subset \mathbb{Z}$ and real coefficients $|\epsilon_k| \leq 1$ is

$$\Lambda_{S, \epsilon}^d(F_0, F_1, F_2) = \sum_{k \in S} \epsilon_k \sum_{\mathbf{I} \in \mathcal{A}_k} L_{\mathbf{I}}(F_0, F_1, F_2). \quad (1.2)$$

Remark (notation). k is an integer index whose domain is determined by whether it indexes dyadic scales or a finite eigenbasis. In Proposition 3.1: In the cubic trace expansion, k ranges over the chosen orthonormal eigenbasis of M , just as i and j do. In Proposition 4.1: The parent scale satisfies $k \geq m + 1$. In Lemma 5.1: J_k is defined for $k \geq m$, and the telescoping identity is asserted for $k \geq m + 1$. In Theorem 1.1: In the statement, k ranges over S ; in the step-function proof the telescoping sum ranges over every integer from $k_- = \min S$ through $k_+ = \max S$, after choosing $m \leq k_- - 1$. $a \oplus b$ is bitwise XOR when its operands are nonnegative integers, whereas $H \oplus K$

is the direct sum when its operands are inner product spaces. For $v \in \mathbb{Z}/3\mathbb{Z}$, $F_v : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is the input evaluated on the cyclic pair (x_v, x_{v+1}) . In Proposition 4.1, Lemma 5.1: The F_v are bounded compactly supported real dyadic step functions at a common fixed scale m ; $F_v(i, j)$ denotes their constant value on the corresponding pair of atoms. In Lemma 5.2: Each F_v belongs to $L^3(\mathbb{R}_+^2)$. In Theorem 1.1: In the statement, the F_v are bounded compactly supported measurable real functions. The proof first treats dyadic step inputs at a common scale m and then passes to general inputs by dyadic step approximation; its completion also applies to arbitrary real $L^3(\mathbb{R}_+^2)$ inputs.

Theorem 1.1. *For every finite $S \subset \mathbb{Z}$ and all real bounded compactly supported measurable functions F_0, F_1, F_2 on $\mathbb{R}_+^2$,*

$$\sum_{k \in S} \sum_{\mathbf{I} \in \mathcal{A}_k} |L_{\mathbf{I}}(F_0, F_1, F_2)| \leq 40 \prod_{v=0}^2 \|F_v\|_{L^3(\mathbb{R}_+^2)}. \quad (1.3)$$

Remark. For $I = [2^k n_I, 2^k(n_I + 1))$, its halves are $I_{\text{left}} = [2^k n_I, 2^k n_I + 2^{k-1})$ and $I_{\text{right}} = [2^k n_I + 2^{k-1}, 2^k(n_I + 1))$.

In particular, (1.2) satisfies the same product bound, uniformly in the scales and coefficients. The absolute values in (1.3) also allow a separate coefficient of modulus at most one for each admissible triple. The estimate extends to all real L^3 inputs by density, as explained in Section 5.

1.1 Context and relation to earlier work

The continuous triangular Hilbert form has the same cyclic arrangement of inputs as (1.1), integrated over $\mathbb{R}^3$ against the principal-value kernel $1/(x_0 + x_1 + x_2)$. Its symmetric L^3 estimate is formulated by Thiele in [3, Section 9, Problem 13]. That contribution also displays the precise dyadic model (1.2), with one coefficient for each scale, and asks for the analogous bound. Theorem 1.1 answers this dyadic question affirmatively and gives the stronger sum of absolute local contributions. The continuous form and its dyadic model are distinct operators; the theorem here is a statement about the latter.

Kovač, Thiele and Zorin-Kranich introduced a dyadic triangular model with coefficients allowed to vary among interval triples [6, Equations (1.5)–(1.6)]. Their Theorem 1.7 gives uniform estimates when one input has either a specified dependence on a Walsh linear combination of its coordinates or a specified fiberwise character structure. Those cases already encompass dyadic versions of Carleson’s operator and uniform bounds for the bilinear Hilbert transform, as explained in their Appendix A. Our estimate allows all three inputs to vary freely. The cyclic operator-trace representation used below also appears in their Equation (1.5).

Another line of work measures cancellation through the growth of truncated norms. Tao obtained sublogarithmic growth for one-dimensional multilinear Hilbert transforms [9, Theorem 1.2]; Zorin-Kranich proved the corresponding sublinear growth in the number of smooth scales for simplex Hilbert transforms [10, Theorem 1.3]. Durcik, Kovač and Thiele then obtained power-type cancellation in both the continuous and dyadic settings [2, Theorems 1 and 4]. For the triangular dyadic form, their bound over m consecutive scales has a factor $m^{1/2}$ at exponents $(4, 4, 2)$; cyclic permutation and multilinear interpolation give the same growth at $(3, 3, 3)$. Theorem 1.1 removes this scale dependence at the symmetric tuple.

Remark (notation). $m \in \mathbb{Z}$ is a chosen common atom scale such that each step input is constant on products of intervals in $\mathcal{D}_m$; it need not be the coarsest such scale. In Theorem 1.1: In the step-function proof with $S \neq \emptyset$, the common atom scale is refined, if necessary, so that $m \leq \min S - 1$.

The strategy of controlling an entangled form by the change of a nonnegative energy has precedents in Kovač’s Bellman-function framework [4, Section 2] and his telescoping identity for

the twisted paraproduct [5, Lemma 2]. These provide a methodological precedent, rather than a bound for the present triangular form. In particular, Kovač explicitly identifies the obstruction to applying that framework to non-bipartite graphs, including the triangle, in [4, Section 6.2]. Here the energy is a trace of a matrix power, and the averaging is over the four children selected by the XOR constraint.

The matrix energy and the dimension-independent trace inequality are taken from the companion manuscript [7, Sections 2–3, Lemmas 2.1, 2.2 and 3.1]; we reproduce their real finite-dimensional proofs. Their underlying matrix calculus is classical: the square-root derivative is a special case of the divided-difference formula in [1, Section 1, Equation (1)], and the order comparison uses monotonicity of the positive square root [8, Theorem, p. 309].

1.2 The energy argument

The proof reduces the form to finite matrices by taking inputs constant on sufficiently small dyadic squares. On a fixed admissible triple, each input becomes a matrix between the atom spaces of its two variables. At each variable, we join its two incident matrices into a single row operator R and assign it the energy $\text{tr}((R^*R)^{3/2})$. Let $e(\mathbf{I})$ be the sum of these three energies, and let $\text{ch}(\mathbf{I})$ consist of the four admissible triples obtained by halving each interval. The key estimate is

$$|L_{\mathbf{I}}(F_0, F_1, F_2)| \leq C_0 2^{2k} \left(\frac{1}{4} \sum_{\mathbf{I}' \in \text{ch}(\mathbf{I})} e(\mathbf{I}') - e(\mathbf{I}) \right), \quad \mathbf{I} \in \mathcal{A}_k, \quad (1.4)$$

with an absolute constant C_0 . At each scale, the weighted total energy $\sum_{\mathbf{I} \in \mathcal{A}_k} 2^{2k} e(\mathbf{I})$ is bounded by an absolute constant times the sum of the cubed input norms.

Remark (notation). $e(\mathbf{I}) = \sum_{v=0}^2 e_v(\mathbf{I})$ is the local matrix energy for $\mathbf{I} \in \mathcal{A}_k$, $k \geq m$, constructed from inputs constant on products of intervals in $\mathcal{D}_m$ using the atom weights $w_v(i) = 2^{m-k}$.

To prove (1.4), a matrix trace estimate bounds the triangular product by three Hessian quadratic forms of $P \mapsto \text{tr}(P^{3/2})$. Conditional on one child half, each neighboring half remains equally likely to be left or right. The two neighbors are correlated, but RR^* is a sum of the two edge Gram matrices; it contains no product between their multipliers and has the same nonzero spectrum as R^*R . Convexity therefore converts the Hessian bound into an increase of the child energy. Writing the local integrals in normalized interval measures leaves the weight 2^{2k} . Each child has one parent and weight $2^{2(k-1)}$, one quarter of the parent's weight, so these increases telescope.

Remark (notation). P is a symmetric positive semidefinite matrix, with $\text{supp } P = (\ker P)^\perp$; zero eigenvalues are omitted when the supported Hessian form is evaluated. In Lemma 2.1: For the smoothness and derivative assertions, P varies among matrices positive definite on a fixed support, and the directions act on that support. In Lemma 2.2: P and Q act on the same space and satisfy $P \leq aQ$.

Section 2 establishes the energy inequalities, Section 3 proves the trace estimate, Section 4 supplies the conditional-averaging argument, and Section 5 sums the resulting differences and passes to measurable inputs.

2 The matrix energy and its curvature

We first record the finite-dimensional facts that will measure the energy increase under a dyadic split. All spaces and matrices in the proof are real; R^* denotes the transpose of R . For a symmetric positive semidefinite matrix P , define

$$\Phi(P) = \text{tr}(P^{3/2}), \quad \text{supp } P = (\ker P)^\perp.$$

A symmetric matrix K is *supported on* P if it annihilates $\ker P$. Symmetry then also puts its range in $\text{supp } P$. For such K , define

$$b_P(K) = \frac{3}{2} \sum_{i,j} \frac{|K_{ij}|^2}{\sqrt{p_i} + \sqrt{p_j}}, \quad (2.1)$$

where $p_i > 0$ are the eigenvalues of P on its support, and the entries are taken in an orthonormal eigenbasis of that support. If $P = 0$, the only supported K is zero and $b_0(0) = 0$. Simultaneous orthogonal conjugation and extension by zero preserve Φ and b .

The next two lemmas are the real-matrix versions of the energy derivative and order comparison results in the companion manuscript [7, Section 2]. We give the proofs, including the support conventions needed for singular Gram matrices.

Lemma 2.1 (Hessian of the energy). *On matrices positive definite on a fixed support, Φ is smooth and, for a symmetric direction K on that support,*

$$d\Phi_P(K) = \frac{3}{2} \text{tr}(\sqrt{P} K), \quad d^2\Phi_P(K, K) = b_P(K).$$

In particular, Φ is convex on the positive semidefinite cone.

Proof. Work on the support and set $B = \sqrt{P}$. The derivative of the squaring map at B is

$$\mathcal{L}_B(X) = BX + XB.$$

This is the square-root instance of the classical matrix derivative calculus [1, Section 1, Equation (1)]. In an eigenbasis of P , this map multiplies entry (i, j) by $\sqrt{p_i} + \sqrt{p_j} > 0$. The inverse function theorem therefore gives a smooth square root whose derivative X in direction K satisfies $BX + XB = K$. Differentiating $\text{tr}(B^3)$ and using cyclicity,

$$d\Phi_P(K) = 3 \text{tr}(B^2 X) = \frac{3}{2} \text{tr}(BK).$$

A second differentiation gives

$$d^2\Phi_P(K, K) = \frac{3}{2} \langle K, \mathcal{L}_B^{-1} K \rangle_{\text{HS}} = b_P(K),$$

where $\langle X, Y \rangle_{\text{HS}} = \text{tr}(X^* Y)$. The Hessian is nonnegative. This proves convexity on positive definite matrices; replacing each matrix by itself plus ηId and letting $\eta \downarrow 0$ proves convexity on the closed cone. $\square$

Lemma 2.2 (Order comparison). *Suppose P, Q are positive semidefinite matrices on the same space, $a \geq 1$, and $P \leq aQ$. Every symmetric K supported on P is supported on Q , and*

$$b_P(K) \geq a^{-1/2} b_Q(K). \quad (2.2)$$

Remark (notation). Q has a local matrix meaning and a separate local nonnegative scalar meaning. In Lemma 2.2: Q is a positive semidefinite matrix on the same space as P , satisfying $P \leq aQ$. In Proposition 3.1: $Q = \sum_{i,j: q_i+q_j>0} \frac{q_i^2 q_j^2}{q_i+q_j} |D_{ij}|^2$ is a nonnegative scalar, where m_i are the eigenvalues of M , $q_i = |m_i|$ are ordered nonincreasingly, and D_{ij} are entries in that orthonormal eigenbasis. a is a real scalar whose meaning is local to its occurrence. In Lemma 2.2: $a \geq 1$ is the comparison factor satisfying $P \leq aQ$. In Lemma 5.1: In the energy estimate, $a, b \geq 0$ are the Hilbert–Schmidt norms of the two edges incident to v ; thus a need not be at least one. In Theorem 1.1: In the inequality $\|a\| - \|b\| \leq \|a - b\|$ used to pass to the absolute sum, a, b are real scalar arguments.

Proof. For $x \in \ker Q$, the inequalities $0 \leq \langle Px, x \rangle \leq a \langle Qx, x \rangle = 0$ imply $Px = 0$. Thus $\ker Q \subseteq \ker P$, which proves the support assertion.

First suppose P, Q are positive definite. Inversion reverses positive definite order: conjugating $0 < X \leq Y$ by $X^{-1/2}$, inverting the resulting matrix at least Id , and conjugating back gives $Y^{-1} \leq X^{-1}$. The spectral integral

$$\sqrt{X} = \frac{1}{\pi} \int_0^\infty X(X + t\text{Id})^{-1} \frac{dt}{\sqrt{t}}$$

then proves monotonicity of the positive square root [8, Theorem, p. 309], since $X(X + t\text{Id})^{-1} = \text{Id} - t(X + t\text{Id})^{-1}$. Consequently $\sqrt{P} \leq \sqrt{a} \sqrt{Q}$.

For a positive semidefinite matrix A , left and right multiplication by A are positive semidefinite self-adjoint operators on the full Hilbert–Schmidt matrix space: their quadratic forms are $\text{tr}(X^*AX)$ and $\text{tr}(X^*XA)$. Hence

$$\mathcal{L}_{\sqrt{P}} \leq \sqrt{a} \mathcal{L}_{\sqrt{Q}}, \quad \mathcal{L}_{\sqrt{P}}^{-1} \geq a^{-1/2} \mathcal{L}_{\sqrt{Q}}^{-1}.$$

Taking the quadratic form at K and using Lemma 2.1 proves (2.2).

In the semidefinite case, $a \geq 1$ gives $P + \eta\text{Id} \leq a(Q + \eta\text{Id})$. Apply the positive definite result and let $\eta \downarrow 0$. In a full eigenbasis of either matrix, every entry of K meeting its kernel is zero. Formula (2.1) therefore gives convergence to the two supported quadratic forms in (2.2). $\square$

Corollary 2.3 (A midpoint gap). *If $T \geq 0$ and $U = U^*$ satisfy $-T \leq U \leq T$, then U is supported on T and*

$$\frac{\Phi(T + U) + \Phi(T - U)}{2} - \Phi(T) \geq \frac{1}{2\sqrt{2}} b_T(U). \quad (2.3)$$

Remark (notation). U is a symmetric matrix on the same space as T . In Corollary 2.3: $-T \leq U \leq T$; consequently U annihilates $\ker T$.

Proof. For $x \in \ker T$, positivity of $T + U$ and $T - U$, whose quadratic forms both vanish at x , implies $Ux = 0$. For $|t| < 1$,

$$(1 - |t|)T \leq T + tU \leq (1 + |t|)T \leq 2T.$$

Thus $T + tU$ has the same support as T . The function $f(t) = \Phi(T + tU)$ satisfies, by Lemmas 2.1 and 2.2,

$$f''(t) = b_{T+tU}(U) \geq 2^{-1/2} b_T(U).$$

Subtracting $b_T(U)t^2/(2\sqrt{2})$ leaves a convex function on $(-1, 1)$, continuous on $[-1, 1]$. Its midpoint inequality is (2.3). If $T = 0$, all terms are zero. $\square$

3 A triangular trace estimate

We now compare a cyclic product of three matrices with the Hessian costs from Section 2. This is the finite-dimensional estimate that will control a local triangular integral. Its proof follows the mixed trace argument in [7, Section 3]; working over the reals lets us identify the two orientations of the triangle directly by transposition.

Let H_v , $v \in \mathbb{Z}/3\mathbb{Z}$, be finite-dimensional real inner product spaces, and let $W_v : H_{v+1} \rightarrow H_v$. The two edges incident to vertex v form the *row star*

$$R_v = [W_v \quad W_{v-1}^*] : H_{v+1} \oplus H_{v-1} \longrightarrow H_v, \quad T_v = R_v^* R_v. \quad (3.1)$$

The two Gram matrices $R_v^* R_v$ and $R_v R_v^*$ have the same nonzero eigenvalues with multiplicities, so their Φ values agree.

Remark (notation). H_v , $v \in \mathbb{Z}/3\mathbb{Z}$, is a finite-dimensional real inner product space. In Proposition 4.1, Lemma 5.1: For $\mathbf{I} \in \mathcal{A}_k$, $k \geq m$, H_v has an orthonormal basis indexed by the intervals of $\mathcal{D}_m$ contained in I_v , so $\dim H_v = 2^{k-m}$. In Proposition 4.1: In the proof, child atom spaces are identified with the corresponding coordinate subspaces of the fixed parent spaces, and child matrices are extended by zero. $W_v : H_{v+1} \rightarrow H_v$ is a real linear map, with vertex indices interpreted in $\mathbb{Z}/3\mathbb{Z}$. In Proposition 4.1, Lemma 5.1: For $\mathbf{I} \in \mathcal{A}_k$, $k \geq m$, the atom-basis entries are $W_v(i, j) = \sqrt{w_v(i)w_{v+1}(j)} F_v(i, j)$, where $F_v(i, j)$ is the input's constant value on the corresponding atom pair. In Proposition 4.1: In the proof, child maps are extended by zero to the parent atom spaces; the extended child edge is $C_v W_v C_{v+1}$, where $C_v = (\text{Id} + s_v D_v)^{1/2}$. $R_v = [W_v \ W_{v-1}^*] : H_{v+1} \oplus H_{v-1} \rightarrow H_v$, with $v \in \mathbb{Z}/3\mathbb{Z}$. In Proposition 4.1, Lemma 5.1: R_v is constructed separately for each admissible triple $\mathbf{I} \in \mathcal{A}_k$, $k \geq m$, from its atom spaces and the weighted edge matrices in equation 4.1. $T_v = R_v^* R_v$ is the Gram operator on $H_{v+1} \oplus H_{v-1}$ of the row star $R_v = [W_v \ W_{v-1}^*]$. In Proposition 4.1, Lemma 5.1: T_v depends on the interval triple through its atom spaces and weighted edges: in $e_v(\mathbf{I}) = \Phi(T_v)$ it is $T_v(\mathbf{I}) = R_v(\mathbf{I})^* R_v(\mathbf{I})$, and a child energy uses the corresponding child's operator.

Proposition 3.1 (Mixed trace estimate). *For the maps in (3.1), let $D_v : H_v \rightarrow H_v$ be symmetric with $\|D_v\|_{\text{op}} \leq 1$, and put $U_v = R_v^* D_v R_v$. Then U_v is supported on T_v , and*

$$|\text{tr}(D_0 W_0 D_1 W_1 D_2 W_2)| \leq \frac{5}{3} \sum_{v=0}^2 b_{T_v}(U_v). \quad (3.2)$$

The constant is independent of all three dimensions.

Remark (notation). $D_v : H_v \rightarrow H_v$, $v \in \mathbb{Z}/3\mathbb{Z}$, is symmetric and satisfies $\|D_v\|_{\text{op}} \leq 1$. In Proposition 4.1: For the parent $\mathbf{I} \in \mathcal{A}_k$, $k \geq m+1$, D_v is diagonal in the atom basis, with diagonal entry equal to the constant value of h_{I_v} on that atom. $U_v = R_v^* D_v R_v$ acts on $H_{v+1} \oplus H_{v-1}$, with $v \in \mathbb{Z}/3\mathbb{Z}$. In Proposition 4.1: For each $\mathbf{I} \in \mathcal{A}_k$ with $k \geq m+1$, U_v uses the row star of that triple and the diagonal matrix D_v of the h_{I_v} -values on its atoms.

Proof. The identity $\ker T_v = \ker R_v$ proves the support assertion. We first place the three quadratic forms on one space, then bound a cubic trace in an eigenbasis.

A common quadratic form. On $H = H_0 \oplus H_1 \oplus H_2$, define

$$M = \begin{pmatrix} 0 & W_0 & W_2^* \\ W_0^* & 0 & W_1 \\ W_2 & W_1^* & 0 \end{pmatrix}, \quad D = \text{diag}(D_0, D_1, D_2).$$

Both matrices are symmetric and $\|D\|_{\text{op}} \leq 1$. Let P_v be the projection onto H_v , and set $A_v = M P_v M$, $K_v = M P_v D M$. The only nonzero block row of $P_v M$ is R_v , with its columns placed in vertex order. Thus A_v and K_v are T_v and U_v , respectively, after a permutation of blocks and extension by zero. In particular K_v is supported on A_v . Also $P_v D = D P_v$, so K_v is symmetric, and

$$0 \leq A_v \leq M^2, \quad \sum_v K_v = M D M.$$

Lemma 2.2, followed by the quadratic-form inequality for a sum of three vectors, gives

$$\mathcal{E} := \sum_v b_{T_v}(U_v) \geq \sum_v b_{M^2}(K_v) \geq \frac{1}{3} b_{M^2}(M D M). \quad (3.3)$$

This comparison is on the relevant supports; M need not be invertible.

The cubic trace. Choose an orthonormal eigenbasis of M , with eigenvalues m_i ordered so that $q_i = |m_i|$ is nonincreasing. Write D_{ij} for the entries of D in this basis. Since $(MDM)_{ij} = m_i m_j D_{ij}$, (3.3) gives

$$\mathcal{E} \geq \frac{1}{2}Q, \quad Q = \sum_{\substack{i,j \\ q_i + q_j > 0}} \frac{q_i^2 q_j^2}{q_i + q_j} |D_{ij}|^2. \quad (3.4)$$

Define

$$A_i = q_i^3 |D_{ii}|^2, \quad E_i = q_i \sum_{j>i} q_j^2 |D_{ij}|^2, \quad A = \sum_i A_i, \quad E = \sum_i E_i.$$

The diagonal terms of Q sum to $A/2$. Pairing its off-diagonal terms and using $q_i \geq q_j$ for $i < j$, we obtain

$$Q = \frac{A}{2} + \sum_{\substack{i<j \\ q_i + q_j > 0}} \frac{2q_i^2 q_j^2}{q_i + q_j} |D_{ij}|^2 \geq \frac{A}{2} + E. \quad (3.5)$$

Expand

$$\text{tr}((DM)^3) = \sum_{i,j,k} m_i m_j m_k D_{ij} D_{jk} D_{ki}.$$

Group the triples by their smallest index i and the number of times it occurs. If it occurs once, its three possible positions give

$$3m_i \sum_{j,k>i} m_j m_k D_{ij} D_{jk} D_{ki} = 3m_i x^* D_{>i} x, \quad x_j = m_j D_{ji}.$$

Here $j = k$ is allowed, and $D_{>i}$ is the compression of D to indices greater than i . This compression is a contraction, so the absolute value is at most $3E_i$.

If the smallest index occurs twice, the contribution is

$$3m_i^2 D_{ii} \sum_{j>i} m_j |D_{ij}|^2.$$

The row norm bound $\sum_j |D_{ij}|^2 \leq 1$ and Cauchy–Schwarz show that its absolute value is at most

$$3q_i^2 |D_{ii}| \left(\sum_{j>i} q_j^2 |D_{ij}|^2 \right)^{1/2} = 3\sqrt{A_i E_i}.$$

When all three indices equal i , the absolute value is at most A_i , since $|D_{ii}| \leq 1$. These cases partition all triples, including repeated indices. Summing and applying Cauchy–Schwarz once more,

$$\begin{aligned} |\text{tr}((DM)^3)| &\leq A + 3E + 3\sqrt{AE} \\ &\leq \frac{5}{2}A + \frac{9}{2}E \leq 5(A/2 + E) \leq 5Q. \end{aligned} \quad (3.6)$$

Zero eigenvalues contribute zero throughout.

Returning to the triangle. In vertex blocks, a nonzero closed three-step product must visit all three vertices because M has zero diagonal blocks. The three starting points in one orientation give $\tau = \text{tr}(D_0 W_0 D_1 W_1 D_2 W_2)$ by cyclicity, and the reverse orientation gives the same real trace by transposition. Thus $\text{tr}((DM)^3) = 6\tau$. Equations (3.4) and (3.6) yield

$$6|\tau| \leq 5Q \leq 10\mathcal{E},$$

which is (3.2). □

4 Energy increase under a dyadic split

We apply the matrix estimate to a fixed admissible triple and compare its energy with the average energy of its children. For this section, the inputs are bounded, compactly supported, and constant on products of intervals in $\mathcal{D}_m$, for some $m \in \mathbb{Z}$.

4.1 Matrices for a local integral

For $\mathbf{I} = (I_0, I_1, I_2) \in \mathcal{A}_k$, $k \geq m$, let H_v have an orthonormal basis indexed by the intervals of $\mathcal{D}_m$ inside I_v . Call these intervals atoms. In the probability measure $|I_v|^{-1} dx_v$, each atom has weight $w_v(i) = 2^{m-k}$. Define $W_v : H_{v+1} \rightarrow H_v$ by

$$W_v(i, j) = \sqrt{w_v(i)w_{v+1}(j)} F_v(i, j), \quad (4.1)$$

where $F_v(i, j)$ is the constant value on the corresponding atom pair. Use these maps in (3.1), and define

$$e_v(\mathbf{I}) = \Phi(T_v) = \Phi(R_v R_v^*), \quad e(\mathbf{I}) = \sum_{v=0}^2 e_v(\mathbf{I}). \quad (4.2)$$

Remark (notation). For $\mathbf{I} \in \mathcal{A}_k$ with $k \geq m$ and $v \in \mathbb{Z}/3\mathbb{Z}$, i indexes an interval of $\mathcal{D}_m$ contained in I_v , and $w_v(i) = 2^{m-k}$ is its mass under $|I_v|^{-1} dx_v$; the 2^{k-m} atom weights sum to one. $e_v(\mathbf{I})$ is defined for $v \in \mathbb{Z}/3\mathbb{Z}$ and $\mathbf{I} \in \mathcal{A}_k$ with $k \geq m$, using the atom spaces and weighted edge matrices at the common atom scale m .

If $k \geq m + 1$, let D_v be the diagonal matrix of the h_{I_v} -values on the atoms, and put

$$\tau(\mathbf{I}) = \text{tr}(D_0 W_0 D_1 W_1 D_2 W_2).$$

The diagonal matrices are symmetric contractions. In the expanded trace, each vertex weight occurs as two square roots, so $\tau(\mathbf{I})$ is exactly the normalized integral of the product of the three inputs and three Haar signs. Therefore

$$L_{\mathbf{I}}(F_0, F_1, F_2) = 2^{2k} \tau(\mathbf{I}). \quad (4.3)$$

Remark (notation). $\tau(\mathbf{I})$ is defined for $\mathbf{I} \in \mathcal{A}_k$ with $k \geq m + 1$, using the weighted atom matrices W_v and the diagonal matrices D_v of the h_{I_v} -values on atoms; equivalently, $\tau(\mathbf{I}) = 2^{-3k} \int_{I_0 \times I_1 \times I_2} F_0(x_0, x_1) F_1(x_1, x_2) F_2(x_2, x_0) \prod_{v=0}^2 h_{I_v}(x_v) dx_0 dx_1 dx_2$.

4.2 The four children

If $b_v \in \{0, 1\}$ chooses the left or right half of I_v , the child index is $2n_{I_v} + b_v$. The child triple is admissible precisely when $b_0 \oplus b_1 \oplus b_2 = 0$. Equivalently, with $s_v = (-1)^{b_v}$, its signs satisfy $s_0 s_1 s_2 = 1$. Write $\text{ch}(\mathbf{I})$ for these four children. Under uniform choice among them, each sign is uniform and every pair of signs is independent. Figure 1 displays the conditional averages used below.

Remark (notation). For a child of $\mathbf{I}$, $b_v = 0$ selects the left half of I_v and $b_v = 1$ selects the right half, whose index is $2n_{I_v} + b_v$; admissible child triples satisfy $b_0 \oplus b_1 \oplus b_2 = 0$.

Proposition 4.1 (Local energy increase). *For $\mathbf{I} \in \mathcal{A}_k$, $k \geq m + 1$, define*

$$\delta(\mathbf{I}) = \frac{1}{4} \sum_{\mathbf{I}' \in \text{ch}(\mathbf{I})} e(\mathbf{I}') - e(\mathbf{I}).$$

Then $\delta(\mathbf{I}) \geq 0$, and

$$|\tau(\mathbf{I})| \leq \frac{10\sqrt{2}}{3} \delta(\mathbf{I}). \quad (4.4)$$

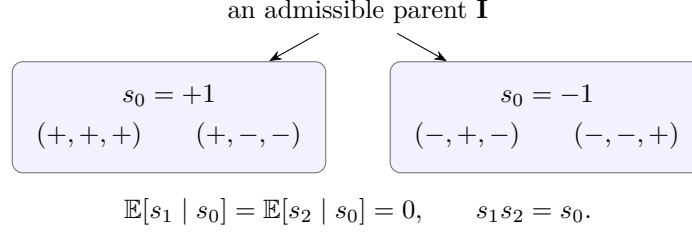


Figure 1: The four admissible child sign triples, grouped by the first sign. Conditional on s_0 , the two possibilities in its box have equal probability. Each neighboring sign is unbiased, but their product is fixed. The sum of the two incident-edge Grams in (4.5) uses only these marginal averages.

Remark (notation). For $\mathbf{I} \in \mathcal{A}_k$ with $k \geq m + 1$, $\delta(\mathbf{I}) = \frac{1}{4} \sum_{\mathbf{I}' \in \text{ch}(\mathbf{I})} e(\mathbf{I}') - e(\mathbf{I})$, where m is the common dyadic step scale of the inputs.

Proof. We keep the parent spaces fixed and extend each child matrix by zero. Set

$$C_v = (\text{Id} + s_v D_v)^{1/2}.$$

It is $\sqrt{2}$ times the projection onto the atoms in the chosen half. Child probability weights double, so the extended child edge matrix is $C_v W_v C_{v+1}$. Its row star is

$$R_v(s) = C_v R_v \text{diag}(C_{v+1}, C_{v-1}).$$

In particular,

$$R_v(s) R_v(s)^* = C_v (W_v C_{v+1}^2 W_v^* + W_{v-1}^* C_{v-1}^2 W_{v-1}) C_v. \quad (4.5)$$

Fix s_v . Each neighboring sign is still uniform, so its squared multiplier has conditional expectation Id . Although the neighbors are correlated, (4.5) has no product between their multipliers. Thus

$$\mathbb{E}[R_v(s) R_v(s)^* | s_v] = C_v R_v R_v^* C_v.$$

By the convexity in Lemma 2.1,

$$\mathbb{E}[e_v(\mathbf{I}(s)) | s_v] \geq \Phi(C_v R_v R_v^* C_v) = \Phi(T_v + s_v U_v), \quad U_v = R_v^* D_v R_v. \quad (4.6)$$

The equality compares the two Gram matrices of $C_v R_v$: their nonzero eigenvalues agree, even though they act on different spaces.

Since D_v is a symmetric contraction, $-T_v \leq U_v \leq T_v$. Averaging (4.6) over $s_v = \pm 1$ and applying Corollary 2.3 gives

$$\frac{1}{4} \sum_{\mathbf{I}' \in \text{ch}(\mathbf{I})} e_v(\mathbf{I}') - e_v(\mathbf{I}) \geq \frac{1}{2\sqrt{2}} b_{T_v}(U_v).$$

Summing over v proves nonnegativity of δ . Proposition 3.1 now gives

$$|\tau(\mathbf{I})| \leq \frac{5}{3} \sum_v b_{T_v}(U_v) \leq \frac{10\sqrt{2}}{3} \delta(\mathbf{I}),$$

as asserted. □

5 Summation and passage to measurable inputs

The local increase from Proposition 4.1 has the sign needed for telescoping toward finer scales. This use of an energy difference parallels the Bellman-function summations in [4, Section 2] and [5, Lemma 2]. We bound the total energy at each level before carrying out that sum.

Lemma 5.1 (Total energy). *For the step functions of Section 4 and every $k \geq m$, set*

$$J_k = \sum_{\mathbf{I} \in \mathcal{A}_k} 2^{2k} e(\mathbf{I}).$$

Then

$$0 \leq J_k \leq 2\sqrt{2} \sum_{v=0}^2 \|F_v\|_3^3. \quad (5.1)$$

For $k \geq m+1$,

$$\sum_{\mathbf{I} \in \mathcal{A}_k} 2^{2k} \delta(\mathbf{I}) = J_{k-1} - J_k. \quad (5.2)$$

Remark (notation). $J_k = \sum_{\mathbf{I} \in \mathcal{A}_k} 2^{2k} e(\mathbf{I})$ is defined for integer $k \geq m$, using inputs constant on products of intervals in $\mathcal{D}_m$. In Theorem 1.1: In the step-function proof with nonempty S , the common atom level is refined so that $m \leq k_- - 1$, where $k_- = \min S$; hence $J_{k_- - 1}$ and all subsequent energies in the telescoping sum are defined.

Proof. If a, b are the Hilbert–Schmidt norms of the two edges incident to v , then

$$e_v(\mathbf{I}) = \text{tr}(T_v^{3/2}) \leq (\text{tr } T_v)^{3/2} = (a^2 + b^2)^{3/2} \leq \sqrt{2}(a^3 + b^3).$$

For an edge W_v , its probability normalization gives

$$\|W_v\|_{\text{HS}}^3 = \left(\frac{1}{|I_v||I_{v+1}|} \int_{I_v \times I_{v+1}} |F_v|^2 \right)^{3/2} \leq \frac{1}{|I_v||I_{v+1}|} \int_{I_v \times I_{v+1}} |F_v|^3. \quad (5.3)$$

Every edge occurs in two row stars. Also, every ordered pair of intervals in $\mathcal{D}_k$ determines exactly one admissible third interval, by XOR. Multiplication by $2^{2k} = |I_v||I_{v+1}|$ and summation of (5.3) therefore prove (5.1). These sums are finite: compact support leaves finitely many nonzero edge matrices, each belonging to one admissible triple.

Deleting the last bit of each child index shows that every admissible child has a unique admissible parent. As $2^{2k}/4 = 2^{2(k-1)}$, the child part in the definition of δ sums to J_{k-1} , proving (5.2). $\square$

Proof of Theorem 1.1 for step functions. The case $S = \emptyset$ is immediate. Otherwise set $k_- = \min S$, $k_+ = \max S$, and refine the common atom level, if necessary, so that $m \leq k_- - 1$. By (4.3), Proposition 4.1, and Lemma 5.1,

$$\begin{aligned} \sum_{k \in S} \sum_{\mathbf{I} \in \mathcal{A}_k} |L_{\mathbf{I}}(F_0, F_1, F_2)| &\leq \frac{10\sqrt{2}}{3} \sum_{k=k_-}^{k_+} (J_{k-1} - J_k) \\ &\leq \frac{10\sqrt{2}}{3} J_{k_- - 1} \leq \frac{40}{3} \sum_{v=0}^2 \|F_v\|_3^3. \end{aligned}$$

The insertion of missing scales is valid because each energy difference is nonnegative. If every input has nonzero norm, apply this estimate to $F_v/\|F_v\|_3$. Trilinearity of each local integral gives (1.3), with constant 40. If any norm is zero, the local integrals vanish. $\square$

To pass to general inputs we need continuity of the absolute sum, not merely continuity of the signed form. The following fixed-level estimate provides it.

Lemma 5.2 (Continuity at one scale). *For every $k \in \mathbb{Z}$ and real $F_v \in L^3(\mathbb{R}_+^2)$,*

$$\sum_{\mathbf{I} \in \mathcal{A}_k} |L_{\mathbf{I}}(F_0, F_1, F_2)| \leq \prod_{v=0}^2 \|F_v\|_3. \quad (5.4)$$

Consequently the sequence $(L_{\mathbf{I}}(F_0, F_1, F_2))_{\mathbf{I} \in \mathcal{A}_k}$ depends continuously on the inputs as a map from $L^3 \times L^3 \times L^3$ to $\ell^1(\mathcal{A}_k)$.

Proof. Define the nonnegative edge averages

$$a_v(\mathbf{I}) = \frac{1}{|I_v||I_{v+1}|} \int_{I_v \times I_{v+1}} |F_v|^3.$$

Hölder's inequality on the product probability space of the three intervals, with the Haar signs bounded by one, gives

$$|L_{\mathbf{I}}(F_0, F_1, F_2)| \leq 2^{2k} \prod_{v=0}^2 a_v(\mathbf{I})^{1/3}.$$

Apply Hölder again, now to the sum over $\mathcal{A}_k$ with weight 2^{2k} . The uniqueness of the third interval yields

$$\sum_{\mathbf{I} \in \mathcal{A}_k} 2^{2k} a_v(\mathbf{I}) = \|F_v\|_3^3,$$

proving (5.4). These bounds also justify all integrals and absolute sums for L^3 inputs.

For two input triples F, G , expand the difference of local integrals into the three terms with inputs

$$(F_0 - G_0, F_1, F_2), \quad (G_0, F_1 - G_1, F_2), \quad (G_0, G_1, F_2 - G_2).$$

Estimate each term by (5.4). If $G_v \rightarrow F_v$ in L^3 , the sum of these three product bounds tends to zero. This is the asserted ℓ^1 continuity. $\square$

Completion of Theorem 1.1. Bounded compactly supported dyadic step functions are dense in $L^3(\mathbb{R}_+^2)$. To see this, first approximate an L^3 function by a continuous compactly supported function, then take its averages on sufficiently small dyadic squares. Uniform continuity and a common bounded support give convergence in L^3 .

For arbitrary real $F_v \in L^3$, choose such step functions $F_v^{(n)} \rightarrow F_v$ in L^3 . Assuming $S \neq \emptyset$, refine their common atom levels, without changing the functions, to satisfy $m_n \leq \min S - 1$. Apply the step-function estimate to these inputs. Lemma 5.2, summed over the fixed finite set S , gives convergence of their local integrals in ℓ^1 . The inequality $||a| - |b|| \leq |a - b|$ permits passage to the absolute sum in (1.3), and norm convergence handles its right-hand side. This proves the theorem and its stated extension to real L^3 inputs. The case $S = \emptyset$ remains immediate.

For any coefficients $|\epsilon_{\mathbf{I}}| \leq 1$, the triangle inequality gives the corresponding bound for

$$\sum_{k \in S} \sum_{\mathbf{I} \in \mathcal{A}_k} \epsilon_{\mathbf{I}} L_{\mathbf{I}}(F_0, F_1, F_2). \quad \square$$

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