

Close Separable C^* -Algebras Without Spatial Conjugacy

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Abstract

We disprove the separable C^* -algebraic spatial form of the Kadison–Kastler conjecture. For every $\varepsilon > 0$, we construct unital, norm-separable C^* -algebras on a common separable complex Hilbert space, with the same identity and Kadison–Kastler distance less than ε , that are not conjugate by any unitary. The two algebras have the same von Neumann closure.

1 Introduction

For C^* -algebras $A, B \subseteq \mathcal{B}(H)$, write A_1 and B_1 for their closed unit balls. Their Kadison–Kastler distance is

$$d_{\text{KK}}(A, B) = \max \left\{ \sup_{a \in A_1} \inf_{b \in B_1} \|a - b\|, \sup_{b \in B_1} \inf_{a \in A_1} \|b - a\| \right\}. \quad (1)$$

The spatial form of the perturbation problem asks whether sufficiently close algebras must be related by $uAu^* = B$ for a unitary $u \in \mathcal{B}(H)$. Kadison and Kastler initiated this problem in [6]. In its separable C^* -algebraic form, one asks for a constant $\delta > 0$ valid for all norm-separable C^* -algebras on separable Hilbert spaces; no bound on $\|u - I\|$ is required. The following theorem resolves negatively this separable C^* -algebraic spatial form of the Kadison–Kastler conjecture, even for unital algebras with a common identity.

Remark (notation). H denotes the common complex Hilbert space on which the algebras under consideration act. In Proposition 2.1, Corollary 2.4: $H = \bigoplus_{n \geq 1} L$, where L is the nonzero separable Hilbert space carrying D . In Theorem 1.1: The constructed witness is $H = \bigoplus_{n \geq 1} L$, with L from Section 4; in particular, H is separable. A denotes the first C^* -algebra in the pair $A, B \subseteq \mathcal{B}(H)$. In Theorem 1.1: A is a unital, norm-separable C^* -algebra with identity I_H ; the proof constructs $A = \mathcal{E}_D(P_0)$ using the inclusions from Section 4. B denotes the second C^* -algebra in the pair $A, B \subseteq \mathcal{B}(H)$. In Theorem 1.1: B is a unital, norm-separable C^* -algebra with identity I_H ; the proof constructs $B = \mathcal{E}_D(P_t)$ using the inclusions from Section 4, with $0 < t < 1$ chosen so that $2c(t) < \varepsilon$.

Theorem 1.1. *For every $\varepsilon > 0$ there are a separable complex Hilbert space H and unital, norm-separable C^* -algebras $A, B \subseteq \mathcal{B}(H)$ with common identity I_H such that*

$$d_{\text{KK}}(A, B) < \varepsilon, \quad A'' = B'',$$

but no unitary $u \in \mathcal{B}(H)$ satisfies $uAu^* = B$.

The scope of this statement is important. Johnson constructed arbitrarily close separable nuclear algebras for which implementing unitaries cannot be chosen close to the identity [5];

those examples are nevertheless spatially conjugate. Choi and Christensen constructed arbitrarily close nonisomorphic algebras [2], but norm nonseparability is essential to those examples. These distinctions, and the resulting separable spatial formulation, are discussed in [3]. In the positive direction, Christensen, Sinclair, Smith, White, and Winter proved spatial conjugacy for sufficiently close separable nuclear C^* -algebras on a separable Hilbert space [3, Theorem 4]; see [4, Theorem 5.4] for the full proof. Theorem 1.1 imposes no nuclearity assumption. It makes no assertion of abstract nonisomorphism and does not give a counterexample for von Neumann algebras.

The construction. We first isolate a way to turn an obstruction for inclusions into an obstruction to arbitrary spatial conjugacy. Let $D \subseteq \mathcal{B}(L)$ be unital, norm-separable, and have scalar center. For unital subalgebras $P, Q \subseteq D$, consider the algebra

$$\mathcal{E}_D(P) = \{(x_n) : x_n \in D, x_n \longrightarrow p \in P \text{ in norm}\}$$

acting diagonally on $\bigoplus_{n \geq 1} L$. Changing P changes only the condition at the limit. The coordinate projections are intrinsic minimal central projections of this algebra. Consequently, a unitary conjugating $\mathcal{E}_D(P)$ onto $\mathcal{E}_D(Q)$ must permute the coordinates and normalize D on every coordinate. Pulling back constant sequences then forces each finite family in Q to be a simultaneous norm limit of families in unitary conjugates of P by normalizers of D .

Remark (notation). L is the complex Hilbert space on which D acts. In Proposition 2.1, Corollary 2.4: L is nonzero and separable. In Theorem 1.1, Lemma 4.2, Lemma 4.3, Lemma 4.5: $L = K \otimes K$, where $K = \ell^2(G)$. $D \subseteq \mathcal{B}(L)$ is the ambient unital C^* -algebra, with identity I_L . In Proposition 2.1, Corollary 2.4: D is norm-separable and $Z(D) = \mathbb{C}I_L$. In Theorem 1.1, Lemma 4.2, Lemma 4.3, Lemma 4.5: $D = C^*(\{\lambda(g) \otimes I_K, I_K \otimes \pi_0(g), I_K \otimes \pi_t(g) : g \in G\} \cup \{I_K \otimes E_{xy} : x, y \in G\}) \subseteq \mathcal{B}(K \otimes K)$, where $E_{xy}\delta_z = \mathbf{1}_{\{z=y\}}\delta_x$.

Write D' for the commutant of D on L , consisting of the bounded operators commuting with every element of D . The obstruction we use is the equality

$$\left\| \sum_i p_i z_i \right\|_{\mathcal{B}(L)} = \left\| \sum_i p_i \otimes z_i \right\|_{\mathcal{B}(L \otimes L)}, \quad p_i \in P, \quad z_i \in D'. \quad (2)$$

This equality is invariant under normalizers of D and passes to simultaneous finite-family norm limits. We construct very close subalgebras $P_0, P_t \subseteq D$ for which it holds for P_0 but fails for P_t . In fact, P_0 and P_t themselves are conjugate by a unitary close to the identity. The obstruction concerns their positions inside D , and the sequence construction makes it an obstruction to every spatial conjugacy of the resulting algebras.

For the close inclusions we use a real-parameter deformation of the regular representation of the free group on countably many generators. This representation-theoretic ingredient belongs to the Pytlik–Szwarc family [7]; its uniform-topology behavior at the regular representation is discussed by Burger, Ozawa, and Thom [1, Theorem 4.6]. We give a direct tree construction and prove the uniform estimate needed here. The additional mechanism is the combination of the relative tensor-norm test with the convergent-sequence reduction. The reduction is stated separately, so it can be used for other pairs of inclusions.

Section 2 proves the reduction. Section 3 constructs the representations, and Section 4 establishes the tensor-norm obstruction. Section 5 assembles the counterexamples. All Hilbert spaces are complex, and all operator tensor products below act on the indicated Hilbert-space tensor products. No abstract choice of C^* -tensor norm is needed. Unital subalgebras are understood to contain the identity of their ambient operator algebra.

2 A convergent-sequence obstruction

We explain how the limit condition records an inclusion $P \subseteq D$. The resulting algebras preserve the distance between limit algebras, but their spatial conjugacies are constrained by the coordinate algebra D .

Proposition 2.1. *Let L be a nonzero separable Hilbert space, let $D \subseteq \mathcal{B}(L)$ be a unital norm-separable C^* -algebra with $Z(D) = \mathbb{C}I_L$, and let $P, Q \subseteq D$ be unital C^* -subalgebras containing I_L . On $H = \bigoplus_{n \geq 1} L$, define*

$$\mathcal{E}_D(P) = \{(x_n)_{n \geq 1} : x_n \in D, x_n \rightarrow p \in P \text{ in norm}\}, \quad (3)$$

and define $\mathcal{E}_D(Q)$ similarly. Then:

- (i) *These are unital norm-separable C^* -algebras with common identity, and*

$$d_{\text{KK}}(\mathcal{E}_D(P), \mathcal{E}_D(Q)) = d_{\text{KK}}(P, Q). \quad (4)$$

- (ii) *If a unitary conjugates $\mathcal{E}_D(P)$ onto $\mathcal{E}_D(Q)$, there are unitaries $U_n \in \mathcal{B}(L)$ with $U_n D U_n^* = D$ such that, for every $b \in Q$, some $p_b \in P$ satisfies*

$$\|b - U_n p_b U_n^*\| \rightarrow 0. \quad (5)$$

The same sequence (U_n) works for every b .

- (iii) *The two algebras have the same commutant, and hence the same von Neumann closure.*

Remark (notation). $(U_n)_{n \geq 1}$ is one sequence of unitaries on L satisfying $U_n D U_n^* = D$ for every n , common to every $b \in Q$: for each such b there is a fixed $p_b \in P$ with $\|b - U_n p_b U_n^*\| \rightarrow 0$. In Proposition 2.1: In the conditional conjugacy argument, for a chosen S satisfying $S \mathcal{E}_D(P) S^* = \mathcal{E}_D(Q)$, $U_n : L \rightarrow L$ is the restriction of S from the n th coordinate to its image coordinate, with both coordinates identified with L . For the chosen sequence $(U_n)_{n \geq 1}$ and each $b \in Q$, $p_b \in P$ is the fixed element $p_b = \lim_{n \rightarrow \infty} U_n^* b U_n$ in operator norm, equivalently $\|b - U_n p_b U_n^*\| \rightarrow 0$; it may depend on b and on the chosen sequence, but not on n .

Proof. A norm-convergent sequence is bounded, so (3) defines a diagonal operator on H , with norm $\sup_n \|x_n\|$. Algebra operations and adjoints preserve the limit condition, and the constant identity belongs to the algebra. To check norm closedness, suppose $x^{(k)} \rightarrow x$ in the supremum norm, with coordinate limits $p_k \in P$. Then

$$\|p_k - p_j\| \leq \sup_n \|x_n^{(k)} - x_n^{(j)}\|.$$

Thus $p_k \rightarrow p \in P$. The estimate

$$\|x_n - p\| \leq \|x_n - x_n^{(k)}\| + \|x_n^{(k)} - p_k\| + \|p_k - p\|$$

shows that $x_n \rightarrow p$. Moreover,

$$\mathcal{E}_D(P) = c_0(\mathbb{N}, D) + \{(p, p, \dots) : p \in P\}. \quad (6)$$

The two summands are norm-separable: P is a subspace of the separable metric space D , and finite-support sequences from a countable dense subset of D are dense in $c_0(\mathbb{N}, D)$. Their sum is norm-separable.

For the metric assertion, write $\alpha(P, Q) = \sup_{p \in P_1} \text{dist}(p, Q_1)$. Given a contraction $x \in \mathcal{E}_D(P)$ with limit p and $\eta > 0$, choose $q \in Q_1$ with $\|p - q\| < \alpha(P, Q) + \eta$, and choose N so that $\sup_{n > N} \|x_n - p\| < \eta$. Keeping the first N entries of x and putting q thereafter gives a contraction $y \in \mathcal{E}_D(Q)$ with $\|x - y\| < \alpha(P, Q) + 2\eta$. This proves the upper bound for the directed distance. Conversely, for a constant contraction $(p, p, \dots)$ and any contraction $y \in \mathcal{E}_D(Q)$ with limit $q \in Q_1$,

$$\|(p, p, \dots) - y\| = \sup_n \|p - y_n\| \geq \|p - q\| \geq \text{dist}(p, Q_1).$$

Taking infima and suprema proves equality of the directed distances. Interchanging P, Q gives (4).

Let e_n be the projection onto the n th copy of L . Every e_n is a central element of $\mathcal{E}_D(P)$. Conversely, a central element (x_n) commutes with all finite-support D -valued sequences, so each x_n lies in $Z(D) = \mathcal{C}I_L$. Thus the center consists exactly of convergent scalar sequences. Every nonzero central projection contains an e_n , and hence the minimal nonzero central projections are precisely the e_n . The same holds for $\mathcal{E}_D(Q)$.

If $S\mathcal{E}_D(P)S^* = \mathcal{E}_D(Q)$, conjugation by S therefore permutes the projections e_n . Its restriction from the n th coordinate to its image is a unitary $U_n : L \rightarrow L$. The corresponding coordinate corners of both algebras are exactly D , since every finitely supported D -valued sequence is allowed. Consequently $U_n D U_n^* = D$. For $b \in Q$, pull back the constant sequence $(b, b, \dots)$ under S . Its n th entry is $U_n^* b U_n$, regardless of the coordinate permutation. It belongs to $\mathcal{E}_D(P)$, so its entries converge in norm to some $p_b \in P$. This is equivalent to (5) and proves (ii).

Finally, an operator commuting with $\mathcal{E}_D(P)$ commutes with all e_n , so it is block diagonal. Commutation with the finite-support D corners means that each block lies in D' . These conditions are also sufficient, and do not depend on P . Thus both commutants equal

$$\{(z_n) : z_n \in D', \sup_n \|z_n\| < \infty\}.$$

This proves (iii). □

We next give a concrete condition preserved by the approximation in Proposition 2.1(ii).

Definition 2.2. For a unital inclusion $P \subseteq D \subseteq \mathcal{B}(L)$, say that P has the *spatial tensor-norm identity relative to D* if

$$\left\| \sum_{i=1}^m p_i z_i \right\|_{\mathcal{B}(L)} = \left\| \sum_{i=1}^m p_i \otimes z_i \right\|_{\mathcal{B}(L \otimes L)} \quad (7)$$

for every positive integer m , all $p_1, \dots, p_m \in P$, and all $z_1, \dots, z_m \in D'$.

Lemma 2.3. *Let $P, Q \subseteq D \subseteq \mathcal{B}(L)$ be unital C^* -algebras. The identity in Definition 2.2 is preserved by conjugating P with any unitary normalizing D . If (U_n) is a sequence of unitaries in $\mathcal{B}(L)$ satisfying $U_n D U_n^* = D$ and every $b \in Q$ admits the approximation (5), then the identity for P implies the identity for Q .*

Proof. If $U D U^* = D$, then $U D' U^* = D'$. For $q_i = U p_i U^*$ and $z_i \in D'$, apply (7) to p_i and $U^* z_i U$. Conjugate its product sum by U and its tensor sum by $U \otimes U$. This gives (7) for q_i, z_i .

Now fix finite families $b_i \in Q$ and $z_i \in D'$. Put $q_{i,n} = U_n p_{b_i} U_n^*$. For each n , the first part gives the identity for the entire list $(q_{i,n})_i$, since its entries all lie in $U_n P U_n^*$. Both errors

$$\left\| \sum_i (q_{i,n} - b_i) z_i \right\|, \quad \left\| \sum_i (q_{i,n} - b_i) \otimes z_i \right\|$$

are at most $\sum_i \|q_{i,n} - b_i\| \|z_i\|$, which tends to zero. Taking the limit proves the identity for the prescribed families in Q and D' . □

Corollary 2.4. *Under the hypotheses of Proposition 2.1, suppose that P has the spatial tensor-norm identity relative to D , whereas Q does not. Then $\mathcal{E}_D(P)$ and $\mathcal{E}_D(Q)$ are not spatially conjugate. If $W \in \mathcal{B}(L)$ is unitary and $WPW^* = Q$, then nevertheless*

$$d_{\text{KK}}(\mathcal{E}_D(P), \mathcal{E}_D(Q)) \leq 2 \|W - I_L\|.$$

Remark (notation). W is a unitary operator on L . In Corollary 2.4: W satisfies $WPW^* = Q$ in the conditional distance assertion and is not required to normalize D . In Lemma 4.2, Theorem 1.1: The supplied choice is $W = W_{\pi_t} W_{\pi_0}^*$, where $W_{\pi_a}(\delta_x \otimes v) = \delta_x \otimes \pi_a(x)v$ for $a = 0, t$; it satisfies $WP_0W^* = P_t$ and $\|W - I_L\| \leq c(t)$.

Proof. A spatial conjugacy would give the unitaries in Proposition 2.1(ii), contradicting Lemma 2.3. For the distance bound, $p \mapsto WPW^*$ bijects the closed unit balls and

$$\|WPW^* - p\| \leq 2 \|p\| \|W - I_L\|.$$

Apply Proposition 2.1(i). □

The unitary W used for the distance estimate is not required to normalize D . Indeed, under the stated tensor-norm obstruction, no unitary normalizing D can conjugate P onto Q .

3 A uniformly small free-group deformation

We now construct the representations used to produce the inclusions in Corollary 2.4. Let G be the free group on the countable set $\{s_1, s_2, \dots\}$, let e denote its identity, and put $K = \ell^2(G)$. On its usual orthonormal basis define

$$\lambda(g)\delta_x = \delta_{gx}, \quad \rho(g)\delta_x = \delta_{xg^{-1}}. \quad (8)$$

Both are unitary representations, and every left translation commutes with every right translation. The group G is countable, so K is separable.

The following elementary version of the real Pytlik–Szwarc deformation [7, 1] records exactly the coefficient and uniform estimate that the proof needs.

Lemma 3.1. *For every $0 < t < 1$, there is a unitary representation $\pi_t : G \rightarrow \mathcal{B}(K)$ such that, with $r = \sqrt{1 - t^2}$,*

$$\langle \pi_t(s_i)\delta_e, \delta_e \rangle = t \quad (i \geq 1), \quad (9)$$

$$\sup_{g \in G} \|\pi_t(g) - \lambda(g)\| \leq c(t), \quad c(t) = 2(1 - r) + \frac{t}{1 - t} + t. \quad (10)$$

In particular, $c(t) \rightarrow 0$ as $t \downarrow 0$.

Remark (notation). t is a real parameter with $0 < t < 1$. In Theorem 1.1: Given $\varepsilon > 0$, t is chosen sufficiently small that $2c(t) < \varepsilon$, where $c(t) = 2(1 - \sqrt{1 - t^2}) + t/(1 - t) + t$.

Proof. Use the tree of reduced words with edges given by right multiplication by a generator or its inverse, rooted at e . Its distance is $d(x, y) = |x^{-1}y|$, so left translations preserve distance. For $x \neq e$, let $p(x)$ be the neighbor of x toward e , obtained by deleting the last letter of its reduced word. Recursively define

$$\xi_e = \delta_e, \quad \xi_x = t\xi_{p(x)} + r\delta_x \quad (x \neq e). \quad (11)$$

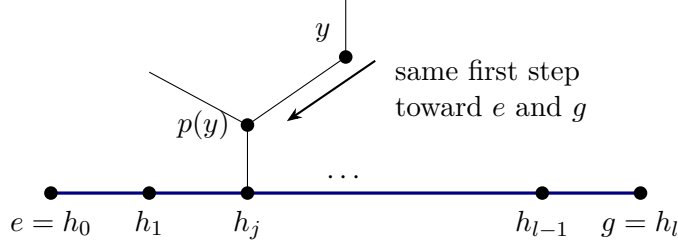


Figure 1: Off the geodesic $[e, g]$, the first edges toward e and g coincide. Hence $\pi_t(g)\lambda(g)^*$ fixes all basis vectors outside the finite path span. Only a finite part of the tree is shown.

These are unit vectors supported on their root paths. If z is the last common vertex of the root paths to x and y , the common-support components are $t^{d(x,z)}\xi_z$ and $t^{d(y,z)}\xi_z$, respectively. The remaining supports are disjoint. Hence

$$\langle \xi_x, \xi_y \rangle = t^{d(x,y)}. \quad (12)$$

Also $\delta_e = \xi_e$ and $\delta_x = (\xi_x - t\xi_{p(x)})/r$ for $x \neq e$, so their span is dense. The assignment $\xi_x \mapsto \xi_{gx}$ preserves every finite Gram matrix by (12). It therefore defines an isometry on the span, and the inverse assignment for g^{-1} makes the extension unitary. Denote this unitary by $\pi_t(g)$. The representation law holds on the dense span and hence on K . Since $\pi_t(s_i)\delta_e = \xi_{s_i} = t\delta_e + r\delta_{s_i}$, (9) follows.

It remains to prove that the perturbation is uniformly small over all group elements, rather than just over the generators. Fix $g \neq e$, write $h_0 = e, h_1, \dots, h_l = g$ for its root path, and put $T = \pi_t(g)\lambda(g)^*$. For $y \neq g$, (11) gives

$$T\delta_y = \frac{\xi_y - t\xi_{gp(g^{-1}y)}}{r}. \quad (13)$$

The vertex $gp(g^{-1}y)$ is the neighbor of y toward g . If y is off the path $[e, g]$, this neighbor is also $p(y)$, the neighbor toward e ; see Figure 1. Thus $T\delta_y = \delta_y$ off that path. On the path, the neighbor toward g of h_k , for $k < l$, is h_{k+1} , so

$$T\delta_{h_k} = r\xi_{h_k} - t\delta_{h_{k+1}} \quad (0 \leq k < l), \quad T\delta_{h_l} = \xi_{h_l}. \quad (14)$$

Let $M_g = \text{span}\{\delta_{h_k} : 0 \leq k \leq l\}$. On this space define the truncated shift $S\delta_{h_k} = \delta_{h_{k+1}}$ for $k < l$ and $S\delta_{h_l} = 0$, and put $X\delta_{h_k} = \xi_{h_k}$. For $R = \text{diag}(1, r, \dots, r)$, the recursion gives

$$X = R(I - tS^*)^{-1}. \quad (15)$$

Indeed, its k th column is $t^k\delta_e + r \sum_{j=1}^k t^{k-j}\delta_{h_j}$. Since $\|S\| \leq 1$ and $\|R\| = 1$,

$$\|X - I\| \leq \|R - I\| + \|R\| \sum_{j=1}^{\infty} t^j \leq 1 - r + \frac{t}{1-t}. \quad (16)$$

Writing $R' = \text{diag}(r, \dots, r, 1)$, (14) becomes $T|_{M_g} = XR' - tS$. Therefore

$$\begin{aligned} \|T|_{M_g} - I\| &\leq \|(X - I)R'\| + \|R' - I\| + t\|S\| \\ &\leq 2(1 - r) + \frac{t}{1-t} + t = c(t). \end{aligned}$$

The same bound holds on K , since T is the identity on $M_g^\perp$. Multiplication by $\lambda(g)$ proves (10). For $g = e$ the difference is zero, so the supremum includes this case as well. $\square$

Remark 3.2. Only the operator X on a single finite path is used in the estimate. On this infinitely branching tree, the global map $\delta_x \mapsto \xi_x$ need not be bounded: the vectors ξ_{s_i} all have root coefficient t . The Gram-matrix construction of π_t does not require such a global bounded map.

4 Close inclusions distinguished by a tensor norm

Fix $0 < t < 1$, let π_t be given by Lemma 3.1, and write $\pi_0 = \lambda$. We construct a common coordinate algebra D and subalgebras P_0, P_t satisfying Corollary 2.4. First we record the absorption calculation that will be used both for closeness and for the tensor-norm estimate.

Lemma 4.1 (Absorption). *Let σ be a unitary representation of G on a nonzero Hilbert space J . The operator*

$$W_\sigma(\delta_x \otimes v) = \delta_x \otimes \sigma(x)v \quad (x \in G, v \in J)$$

is unitary on $K \otimes J$ and satisfies

$$W_\sigma(\lambda(g) \otimes I_J)W_\sigma^* = \lambda(g) \otimes \sigma(g) \quad (g \in G). \quad (17)$$

Remark (notation). σ is a unitary representation of G on a nonzero Hilbert space. In Lemma 4.5: $\sigma = \pi_t \otimes \rho \otimes 1$ acts on $K^{\otimes 3}$, where 1 is the trivial representation on the last copy of K .

Proof. The operator is the orthogonal direct sum of the unitaries $\sigma(x)$. On $\delta_x \otimes v$, the left side of (17) equals

$$\delta_{gx} \otimes \sigma(gx)\sigma(x)^*v = \delta_{gx} \otimes \sigma(g)v. \quad \square$$

On $L = K \otimes K$, put

$$Q_a(g) = \lambda(g) \otimes \pi_a(g), \quad P_a = C^*(Q_a(g) : g \in G) \quad (a = 0, t). \quad (18)$$

For the matrix units $E_{xy}\delta_z = \mathbf{1}_{\{z=y\}}\delta_x$ on K , define $D \subseteq \mathcal{B}(L)$ to be generated by

$$\lambda(g) \otimes I_K, \quad I_K \otimes \pi_0(g), \quad I_K \otimes \pi_t(g) \quad (g \in G), \quad I_K \otimes E_{xy} \quad (x, y \in G). \quad (19)$$

The generators include the identity, and $P_0, P_t \subseteq D$. All three algebras are norm-separable, since their generating sets are countable.

Lemma 4.2. *The algebra D in (19) has scalar center, and*

$$D' = \{Y \otimes I_K : Y \in \lambda(G)'\}. \quad (20)$$

There is a unitary $W \in \mathcal{B}(L)$ such that

$$WP_0W^* = P_t, \quad \|W - I_L\| \leq c(t). \quad (21)$$

Proof. Commutation with all $I_K \otimes E_{xy}$ forces a bounded operator to be $Y \otimes I_K$: commutation with the diagonal matrix units removes off-diagonal blocks, and commutation with the remaining units makes the diagonal blocks identical. The first-factor generators then impose exactly $Y \in \lambda(G)'$. Such operators commute with all the second-factor generators, proving (20). In particular, $\rho(g) \otimes I_K \in D'$.

If $Y \otimes I_K \in Z(D)$, then Y commutes with $\lambda(G)$ by (20). It also commutes with $\rho(G)$, since $Y \otimes I_K$ belongs to D and the right translations belong to D' . Thus $Y\delta_e$ is invariant under $\lambda(g)\rho(g)$, which acts on the basis by conjugation $x \mapsto gxg^{-1}$. Every nonidentity conjugacy class in G is infinite: conjugate a fixed nonidentity reduced word by distinct generators absent from that word.

A square-summable vector invariant under conjugation must therefore be supported at e . Write $Y\delta_e = \alpha\delta_e$. Commuting with left translations gives $Y\delta_x = \alpha\delta_x$ for every x , proving $Z(D) = \mathbb{C}I_L$.

For the final assertion, Lemma 4.1 shows that $W = W_{\pi_t}W_{\pi_0}^*$ sends every $Q_0(g)$ to $Q_t(g)$ by conjugation. Moreover, the direct-sum description of these unitaries gives

$$\|W - I_L\| = \|W_{\pi_t} - W_{\pi_0}\| = \sup_{x \in G} \|\pi_t(x) - \lambda(x)\| \leq c(t). \quad \square$$

Lemma 4.3. *The inclusion $P_0 \subseteq D$ has the spatial tensor-norm identity.*

Proof. Define a unitary on L by

$$V(v \otimes \delta_x) = \lambda(x)^* v \otimes \delta_x.$$

It commutes with every element of D' by (20). On elementary vectors one computes

$$VQ_0(g)V^*(v \otimes \delta_x) = v \otimes \delta_{gx}.$$

Thus $VP_0V^* = I_K \otimes C^*(\lambda(G))$. Given finite families $p_i \in P_0$ and $z_i \in D'$, write

$$Vp_iV^* = I_K \otimes X_i, \quad z_i = Y_i \otimes I_K.$$

Conjugation by V gives

$$\left\| \sum_i p_i z_i \right\| = \left\| \sum_i Y_i \otimes X_i \right\|.$$

On $L \otimes L = K^{\otimes 4}$, conjugation by $V \otimes I_L$ gives

$$\left\| \sum_i p_i \otimes z_i \right\| = \left\| \sum_i I_K \otimes X_i \otimes Y_i \otimes I_K \right\| = \left\| \sum_i Y_i \otimes X_i \right\|.$$

The last equality permutes the four tensor factors and uses the fact that tensoring with a nonzero identity operator does not change norm. This proves (7). $\square$

For P_t , the identity will fail because the product norm has a lower bound linear in the number of generators, while absorption gives a square-root upper bound on the tensor norm.

Lemma 4.4. *For every positive integer m ,*

$$\left\| \sum_{i=1}^m \lambda(s_i) \right\| \leq 2\sqrt{m}. \quad (22)$$

Proof. Let F_i be the orthogonal projection onto the closed span of words beginning with s_i^{-1} . The F_i are pairwise orthogonal. Write $\lambda(s_i) = A_i + B_i$, where

$$A_i = \lambda(s_i)(I_K - F_i), \quad B_i = \lambda(s_i)F_i.$$

There is no cancellation when s_i is prepended to a word not beginning with s_i^{-1} . Thus the range of A_i consists of vectors supported on words beginning with s_i , and these ranges are mutually orthogonal. For every $v \in K$,

$$\left\| \sum_i A_i v \right\|^2 = \sum_i \|(I_K - F_i)v\|^2 \leq m \|v\|^2,$$

while

$$\left\| \sum_i B_i v \right\| \leq \sum_i \|F_i v\| \leq \sqrt{m} \left(\sum_i \|F_i v\|^2 \right)^{1/2} \leq \sqrt{m} \|v\|.$$

Adding the two bounds proves (22). $\square$

Lemma 4.5. *For every $0 < t < 1$, the inclusion $P_t \subseteq D$ fails the spatial tensor-norm identity.*

Proof. Choose an integer $m > 4/t^2$, so $mt > 2\sqrt{m}$, and take

$$p_i = Q_t(s_i), \quad z_i = \rho(s_i) \otimes I_K, \quad 1 \leq i \leq m.$$

These belong to P_t and D' , respectively. Since $\lambda(s_i)\rho(s_i)\delta_e = \delta_e$, (9) implies

$$\left\| \sum_i p_i z_i \right\| \geq \left| \left\langle \left(\sum_i p_i z_i \right) (\delta_e \otimes \delta_e), \delta_e \otimes \delta_e \right\rangle \right| = mt. \quad (23)$$

On the other hand, the tensor sum on $L \otimes L$ is

$$\sum_{i=1}^m \lambda(s_i) \otimes \pi_t(s_i) \otimes \rho(s_i) \otimes I_K.$$

Apply Lemma 4.1 to the representation $\sigma = \pi_t \otimes \rho \otimes 1$ on $K^{\otimes 3}$, where 1 is the trivial representation on the last copy of K . Together with Lemma 4.4, it gives

$$\left\| \sum_i p_i \otimes z_i \right\| = \left\| \sum_{i=1}^m \lambda(s_i) \right\| \leq 2\sqrt{m} < mt. \quad (24)$$

Equations (23) and (24) contradict the identity for this finite list. $\square$

5 The counterexamples

Proof of Theorem 1.1. Given $\varepsilon > 0$, choose $0 < t < 1$ so small that $2c(t) < \varepsilon$. Use the algebras $P_0, P_t \subseteq D \subseteq \mathcal{B}(L)$ from Section 4, and put

$$H = \bigoplus_{n \geq 1} L, \quad A = \mathcal{E}_D(P_0), \quad B = \mathcal{E}_D(P_t).$$

The space H is separable. Lemma 4.2 supplies the scalar center of D , so Proposition 2.1 shows that A, B are unital norm-separable C^* -algebras with common identity and that $A'' = B''$. The same lemma supplies a unitary W with $WP_0W^* = P_t$ and $\|W - I_L\| \leq c(t)$. By Corollary 2.4,

$$d_{\text{KK}}(A, B) \leq 2c(t) < \varepsilon.$$

Finally, Lemmas 4.3 and 4.5 show that the relative tensor-norm identity holds for P_0 but not for P_t . Corollary 2.4 therefore excludes every unitary conjugating A onto B . $\square$

The common ideal $c_0(\mathbb{N}, D)$ makes the von Neumann closures agree, while the norm-limit condition retains the distinction between the two inclusions.

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