

An Exact Semidefinite Lift of a Nonspectrahedral Hyperbolicity Cone

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Abstract

We construct an exact semidefinite lift of the explicit nonspectrahedral hyperbolicity cone in twenty-three variables defined in the companion paper. The lift is a homogeneous real symmetric pencil of size 100 with 307 auxiliary variables and represents the entire closed cone, including every point with singular X . Thus, although this cone has no semidefinite representation in its original coordinates, it admits one when auxiliary variables are allowed. The stated sizes are not claimed to be minimal.

1 Introduction

Let $\mathbb{S}^n$ denote the real symmetric $n \times n$ matrices. A *spectrahedron* is a set defined by a finite affine linear matrix inequality; a *spectrahedral shadow* is a linear projection of such a set. Thus auxiliary variables may make a semidefinite representation possible even when no representation exists in the original variables. We study this distinction for one explicit hyperbolicity cone.

Remark (notation). In the matching calculation, $n = N_1 \in \mathbb{C}^{1 \times 2}$ is the first row of $N = SR_0$, and $n^M = nM$.

For a homogeneous real polynomial p with $p(e) > 0$, hyperbolicity with respect to e means that $t \mapsto p(te - x)$ has only real roots for every x . We use the closed-cone convention

$$K(p, e) = \{x : \text{every root of } t \mapsto p(te - x) \text{ is nonnegative}\}.$$

In particular, zero roots are allowed.

Remark (notation). p is a homogeneous real polynomial on its ambient real vector space, considered with a direction e satisfying $p(e) > 0$. In Theorem 1.1, Lemma 2.2, Proposition 4.1: $p(X, Z, y) = \det((\det X)Z - \Phi_y(\text{adj } X))$ on $\mathcal{V} = \mathbb{S}^4 \times \mathbb{S}^4 \times \mathbb{R}^3$; it has degree twenty and satisfies $p(I_4, I_4, 0) = 1$. The symbol e denotes the direction with $p(e) > 0$ used to define hyperbolicity and the closed cone $K(p, e)$ for the polynomial under consideration. In Theorem 1.1, Lemma 2.2, Proposition 4.1: $e = (I_4, I_4, 0) \in \mathcal{V}$ is the specified direction for the polynomial in (3), and $p(e) = 1$. x is a variable point in the ambient space of p . In Theorem 1.1, Lemma 2.2: $x = (X, Z, y) \in \mathcal{V} \cong \mathbb{R}^{23}$, with coordinates given by the upper triangular entries of X , then those of Z , each in lexicographic order of (i, j) , followed by y_1, y_2, y_3 . t is local real notation, with its scalar or vector role determined by the expression in which it occurs. In Lemma 2.2: $t \in \mathbb{R}$ is the scalar indeterminate in $p(te - (\mathcal{X}, \mathcal{Z}, Y_\theta(\eta, \xi))) = \det(tI_{20} - L_\theta(\mathcal{X}, \mathcal{Z}, \eta, \xi))$. In Proposition 4.1: In $t^T H(\Lambda) t$, $t = (t_{i,a}) \in \mathbb{R}^{100}$ collects the real coefficients in $q_i = \sum_{a=1}^5 t_{i,a} f_a$, $1 \leq i \leq 20$. In the later test $D = tvv^T$ for $v \in \ker X$, t is instead an arbitrary scalar ranging over all of $\mathbb{R}$.

The convexity of hyperbolicity cones goes back to Gårding [Går59]. In three variables, the Helton–Vinnikov theorem yields definite symmetric determinantal representations, and hence spectrahedral

cones [HV07, LPR05]. Allowing auxiliary variables leads to a broader representation question. Netzer and Sanyal proved semidefinite liftability when every nonzero boundary point is a smooth point of the defining hyperbolic polynomial [NS15]. More recently, Scheiderer obtained second-order cone lifts under a Nash-smooth boundary hypothesis [Sch25]. Our result instead concerns an explicit cone and gives its lift by a bounded-degree algebraic construction.

1.1 The cone and the main result

For $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, define

$$Q(y) = \begin{pmatrix} y_1^2 + y_2^2 & -y_1y_2 & -y_1y_3 \\ -y_1y_2 & y_2^2 + y_3^2 & -y_2y_3 \\ -y_1y_3 & -y_2y_3 & y_3^2 + y_1^2 \end{pmatrix}. \quad (1)$$

The associated biquadratic form $z^T Q(y) z$ is the coefficient-one Choi–Lam form [CL77, Section 4]. For $a \in \mathbb{R}$, $r \in \mathbb{R}^3$, and $T \in \mathbb{S}^3$, set

$$\Phi_y \begin{pmatrix} a & r^T \\ r & T \end{pmatrix} = \begin{pmatrix} \text{tr}(Q(y)T) & -r^T Q(y) \\ -Q(y)r & aQ(y) \end{pmatrix}. \quad (2)$$

This is a linear map from $\mathbb{S}^4$ to itself for each fixed y . On the space $\mathcal{V} = \mathbb{S}^4 \times \mathbb{S}^4 \times \mathbb{R}^3$, define

$$p(X, Z, y) = \det((\det X)Z - \Phi_y(\text{adj } X)), \quad e = (I_4, I_4, 0), \quad (3)$$

where $\text{adj } X$ denotes the classical adjugate. Each matrix entry inside this determinant is homogeneous of degree five, and $p(e) = 1$; hence p is homogeneous of degree twenty. We write $K = K(p, e)$. The slice representation in Section 2 will in particular verify that p is hyperbolic with respect to e .

Remark (notation). For $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, $Q(y) = \begin{pmatrix} y_1^2 + y_2^2 & -y_1y_2 & -y_1y_3 \\ -y_1y_2 & y_2^2 + y_3^2 & -y_2y_3 \\ -y_1y_3 & -y_2y_3 & y_3^2 + y_1^2 \end{pmatrix} \in \mathbb{S}^3$. In

Lemma 3.1, Proposition 4.1: The two-argument notation denotes the symmetric bilinear polarization $Q(y, z) = \frac{1}{2}(Q(y+z) - Q(y) - Q(z))$ for $y, z \in \mathbb{R}^3$, so $Q(y, y) = Q(y)$. $z \in \mathbb{R}^3$ is a real vector argument of Q and its bilinear polarization. In Lemma 3.1: $(z_1, z_2) \neq (0, 0)$, and the proof writes $z = \tau d(\alpha) + \rho k$ with $\tau > 0$. In Proposition 4.1: The matching construction initially uses $(z_1, z_2) \neq (0, 0)$. The scalar inequality (23) is then extended to every $z \in \mathbb{R}^3$ by continuity and subsequently specialized to $z = y$; no extension of C is asserted. a is local notation whose scalar or index meaning is determined by the displayed construction in which it occurs. In Theorem 1.1, Lemma 2.2, Proposition 4.1: In the definition of Φ_y , $a \in \mathbb{R}$ is the upper-left entry of its matrix argument. In Lemma 3.1: In the rotation construction, $a = a(\theta) \in \mathbb{C}$ is $a = (|g|^2 + |h|^2 \kappa)/\delta$ when $\delta > 0$, and $a = 1$ when $\delta = 0$, where $\delta = |g|^2 + |h|^2$ and $\kappa(\theta) = (w_0/w)^2$. In Theorem 1.1, Proposition 4.1: In $H(\Lambda)_{(i,a),(j,b)}$, a is an integer basis index with $1 \leq a \leq 5$.

Identify $\mathcal{V}$ with $\mathbb{R}^{23}$ by listing the upper triangular entries of X , then those of Z , each in lexicographic order of (i, j) , followed by y_1, y_2, y_3 .

Theorem 1.1. *There exist fixed matrices $A_1, \dots, A_{23}, B_1, \dots, B_{307} \in \mathbb{S}^{100}$ such that, for every $x = (X, Z, y) \in \mathcal{V}$,*

$$x \in K \iff \text{there exists } v \in \mathbb{R}^{307} \text{ with } \sum_{\nu=1}^{23} x_\nu A_\nu + \sum_{j=1}^{307} v_j B_j \succeq 0.$$

The equivalence holds on the entire closed cone, including points with singular X .

Remark. In the moment-matrix definition, $(f_1, \dots, f_5) = (1, \cos \theta, \sin \theta, \cos 2\theta, \sin 2\theta)$ are real functions and $H(\Lambda) \in \mathbb{S}^{100}$ is indexed by (i, a) with $1 \leq i \leq 20$ and $1 \leq a \leq 5$.

Remark (notation). $A_\nu \in \mathbb{S}^{100}$, $1 \leq \nu \leq 23$, are fixed coefficient matrices, independent of the output point x and the auxiliary vector. The symbol v denotes a locally bound vector, with its space and meaning determined by the enclosing formula. In Theorem 1.1: The vector $v \in \mathbb{R}^{307}$ is existentially chosen for each output point x in the displayed semidefinite representation. In Lemma 3.1: In the proof, $v = N_3 - iN_2 \in \mathbb{C}^{1 \times 2}$, and subscripts $+$ and $-$ denote its first and second components. In Proposition 4.1: In the range-condition argument, v is an arbitrary vector in $\ker X \subseteq \mathbb{R}^4$.

The sizes in Theorem 1.1 are explicit bounds, with no minimality assertion. The matrices are specified by a finite coefficient rule in Section 4.

The cone in (3) was constructed in the companion manuscript [Ope26], whose Theorem 1.1 proves that it has no homogeneous semidefinite representation in its original twenty-three coordinates. This also excludes affine representations. Indeed, if a cone containing 0 equals $\{x : A_0 + L(x) \succeq 0\}$, where L is linear, then $A_0 \succeq 0$. For each point x of the cone and $t > 0$, we have $A_0/t + L(x) \succeq 0$; letting $t \rightarrow \infty$ gives $L(x) \succeq 0$. Conversely, $L(x) \succeq 0$ implies $A_0 + L(x) \succeq 0$; thus the same cone has the homogeneous representation $L(x) \succeq 0$. We use the companion result only for this comparison: the proof of Theorem 1.1 below is self-contained. The companion also establishes hyperbolicity and, in Proposition 3.2, describes the part of K with $X \succ 0$. Our slice calculation recovers those two facts as part of the lift construction.

1.2 Proof strategy

The starting point is a family of planes in $\mathbb{R}^3$:

$$y = Y_\theta(\eta, \xi) = \eta(\cos \theta, \sin \theta, 0) + \xi(0, 0, 1).$$

They cover all values of y . On each plane, the matrix $Q(y)$ has a Gram factorization linear in (η, ξ) and trigonometric of degree three in θ . This gives a 20×20 linear pencil L_θ whose positive semidefinite locus is exactly the corresponding slice of K , including its boundary.

The angular parameter does not itself furnish a finite linear matrix inequality. We instead introduce a real linear functional on a finite-dimensional space of trigonometric polynomials with coefficients linear in the slice variables. We require it to be nonnegative on $q^T L_\theta q$ for every vector q of trigonometric polynomials of degree at most two. These requirements form one finite matrix inequality. Evaluation at a single angle provides a feasible functional for every point of K .

Remark (notation). q is a real twenty-component trigonometric polynomial vector in $(\mathcal{T}_2)^{20}$. In Proposition 4.1: The constraint (21) ranges over every $q \in (\mathcal{T}_2)^{20}$. To derive (23), choose $q(\theta) = ((-DJ_u^T b_1(\theta))^T, \dots, (-DJ_u^T b_4(\theta))^T, u^T)^T$, where $b_j(\theta)$ are the columns of the realification $\tilde{C}(\theta)$ supplied by Lemma 3.1 with $S = J_u D J_u^T$ and $(z_1, z_2) \neq (0, 0)$.

This use of a finite matrix to encode positivity against a space of test polynomials belongs to the moment and sum-of-squares approach to semidefinite representations [Las01, Las09]. Here exactness must hold at one fixed degree.

The main issue is the converse: an arbitrary feasible functional need not be evaluation at an angle, or integration against a measure. The degree-two matching identity in Section 3 supplies enough test vectors to recover the matrix inequalities defining each slice. Its construction rotates a fixed complex Gram factor by a 2×2 unitary matrix. The rotation preserves the Gram matrix and reduces a cubic trigonometric pairing to a first harmonic. Matching its value and first derivative then identifies the pairing exactly. This bounded-degree construction is the principal additional algebraic ingredient of the lift. Section 4 applies it to prove exactness for every feasible functional and completes the proof of Theorem 1.1.

2 Exact semidefinite representations on angular slices

We first construct a matrix pencil on each plane $y = Y_\theta(\eta, \xi)$. Its determinant will agree with p on that plane. This will identify cone membership with positive semidefiniteness even when X is singular.

For $\theta \in \mathbb{R}$, set

$$c = \cos \theta, \quad s = \sin \theta, \quad w = e^{i\theta}, \quad d(\theta) = (c, s, 0), \quad k = (0, 0, 1),$$

and write

$$Y_\theta(\eta, \xi) = \eta d(\theta) + \xi k \quad (\eta, \xi \in \mathbb{R}).$$

These planes cover $\mathbb{R}^3$. We first factor Q on each plane, using complex notation to keep the formulas short. For a complex matrix $F = (f_1, f_2) \in \mathbb{C}^{3 \times 2}$, define its realification by

$$\tilde{F} = (\operatorname{Re} f_1, \operatorname{Im} f_1, \operatorname{Re} f_2, \operatorname{Im} f_2).$$

Thus $\tilde{F}\tilde{F}^T = \operatorname{Re}(FF^*)$, where $*$ denotes conjugate transpose. Define

$$U(\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} -w^{-1} & -w \\ sw^{-2} & sw^2 \\ cw^{-2} & -cw^2 \end{pmatrix}, \quad V_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ i & i \\ 1 & 1 \end{pmatrix}, \quad F_\theta(\eta, \xi) = \eta U(\theta) + \xi V_0. \quad (4)$$

Remark (notation). The notation f_1 is local to its defining tuple: it is the first complex column in $F = (f_1, f_2)$ and the constant real function $f_1(\theta) = 1$ in the trigonometric basis $(f_1, \dots, f_5)$. The notation f_2 is local to its defining tuple: it is the second complex column in $F = (f_1, f_2)$ and the real function $f_2(\theta) = \cos \theta$ in the trigonometric basis $(f_1, \dots, f_5)$.

Lemma 2.1 (Angular Gram factorization). *For every $\theta, \eta, \xi \in \mathbb{R}$,*

$$\tilde{F}_\theta(\eta, \xi) \tilde{F}_\theta(\eta, \xi)^T = Q(Y_\theta(\eta, \xi)). \quad (5)$$

In particular, $Q(y) \succeq 0$ for every $y \in \mathbb{R}^3$.

Proof. The three coefficients in the quadratic expansion of $\operatorname{Re}(F_\theta F_\theta^*)$ are

$$\begin{aligned} \operatorname{Re}(UU^*) &= \begin{pmatrix} 1 & -sc & 0 \\ -sc & s^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}, & \operatorname{Re}(V_0 V_0^*) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \operatorname{Re}(UV_0^* + V_0 U^*) &= \begin{pmatrix} 0 & 0 & -c \\ 0 & 0 & -s \\ -c & -s & 0 \end{pmatrix}. \end{aligned}$$

For the (2,3) entry of the last matrix, the multiplication gives $s \cos(2\theta) - c \sin(2\theta) = -s$. The sum of these matrices with coefficients $\eta^2, \xi^2, \eta\xi$ is exactly $Q(\eta c, \eta s, \xi)$. Every real y has this form, so the factorization also proves positive semidefiniteness. $\square$

2.1 The slice pencil

For a real column $r \in \mathbb{R}^3$ and a real vector $u = (u_0, u') \in \mathbb{R} \times \mathbb{R}^3$, define

$$B(r) = \begin{pmatrix} 0 & r^T \\ -r & 0_{3 \times 3} \end{pmatrix}, \quad J_u = \begin{pmatrix} u' & -u_0 I_3 \end{pmatrix}. \quad (6)$$

Then $B(r)u = J_u^T r$. If $r_1, \dots, r_4$ are the columns of $\tilde{F}_\theta(\eta, \xi)$, direct multiplication and Lemma 2.1 give

$$\sum_{j=1}^4 B(r_j)^T W B(r_j) = \Phi_{Y_\theta(\eta, \xi)}(W) \quad (W \in \mathbb{S}^4). \quad (7)$$

Indeed, for $W = \begin{pmatrix} a & v^T \\ v & T \end{pmatrix}$, the individual summands have blocks $r_j^T T r_j$, $-r_j^T v r_j^T$, $-r_j r_j^T v$, and $a r_j r_j^T$. Summing uses $\sum_j r_j r_j^T = Q(Y_\theta(\eta, \xi))$. Consequently, for every $y \in \mathbb{R}^3$,

$$u^T \Phi_y(W) u = \text{tr}(J_u W J_u^T Q(y)). \quad (8)$$

Here we used that the angular planes cover $\mathbb{R}^3$.

Remark (notation). For $1 \leq j \leq 4$, $r_j \in \mathbb{R}^3$ denotes the j th column of the realification of $F_\theta(\eta, \xi)$ at the slice parameters currently in use. In Proposition 4.1: In the calculation leading to (23), r_j is the j th column of $\tilde{F}_\theta(\eta, \xi)$ as a function of the slice variables. After choosing $y = Y_{\theta_0}(\eta_0, \xi_0)$, it denotes the j th column of $\tilde{F}_{\theta_0}(\eta_0, \xi_0)$ for the remainder of the proof, with $B_j = B(r_j)$.

Write $\mathcal{X}, \mathcal{Z} \in \mathbb{S}^4$ for the matrix variables on a slice, and define

$$L_\theta(\mathcal{X}, \mathcal{Z}, \eta, \xi) = \begin{pmatrix} I_4 \otimes \mathcal{X} & \mathcal{B}_\theta(\eta, \xi) \\ \mathcal{B}_\theta(\eta, \xi)^T & \mathcal{Z} \end{pmatrix}, \quad \mathcal{B}_\theta(\eta, \xi) = \begin{pmatrix} B(r_1) \\ B(r_2) \\ B(r_3) \\ B(r_4) \end{pmatrix}. \quad (9)$$

For fixed θ , this is a homogeneous real symmetric linear pencil of size twenty. Its coefficients are trigonometric polynomials in θ of degree at most three.

Lemma 2.2. *For every $\theta, \eta, \xi \in \mathbb{R}$ and $\mathcal{X}, \mathcal{Z} \in \mathbb{S}^4$,*

$$(\mathcal{X}, \mathcal{Z}, Y_\theta(\eta, \xi)) \in K \iff L_\theta(\mathcal{X}, \mathcal{Z}, \eta, \xi) \succeq 0. \quad (10)$$

In particular, p is hyperbolic with respect to e . If $\mathcal{X} \succ 0$, the conditions in (10) are also equivalent to $\mathcal{Z} \succeq \Phi_{Y_\theta(\eta, \xi)}(\mathcal{X}^{-1})$.

Proof. For invertible $\mathcal{X}$, the block determinant formula and (7) show that

$$\begin{aligned} \det L_\theta &= (\det \mathcal{X})^4 \det(\mathcal{Z} - \Phi_{Y_\theta(\eta, \xi)}(\mathcal{X}^{-1})) \\ &= p(\mathcal{X}, \mathcal{Z}, Y_\theta(\eta, \xi)). \end{aligned}$$

The second equality uses $\text{adj } \mathcal{X} = (\det \mathcal{X}) \mathcal{X}^{-1}$ and linearity of Φ_y in its matrix argument. Both sides are polynomials in the slice variables, so the identity also holds for singular $\mathcal{X}$. Moreover, $L_\theta(I_4, I_4, 0, 0) = I_{20}$. Thus

$$p(te - (\mathcal{X}, \mathcal{Z}, Y_\theta(\eta, \xi))) = \det(tI_{20} - L_\theta(\mathcal{X}, \mathcal{Z}, \eta, \xi)). \quad (11)$$

The roots are the real eigenvalues of this symmetric pencil evaluated at the specified point. Since every y lies on some angular plane, this proves hyperbolicity. Nonnegativity of all these eigenvalues, including zero eigenvalues, proves (10). For $\mathcal{X} \succ 0$, the final assertion is the Schur complement criterion applied to (9). $\square$

3 A degree-two matching identity

Write

$$Q(y, z) = \frac{1}{2}(Q(y + z) - Q(y) - Q(z))$$

for the symmetric bilinear polarization of Q , so that $Q(y, y) = Q(y)$.

For $S \in \mathbb{S}^3$ and $F, H \in \mathbb{C}^{3 \times 2}$, put

$$\langle F, H \rangle_S = \operatorname{Re} \operatorname{tr}(F^* S H) = \operatorname{tr}(\tilde{F}^T S \tilde{H}).$$

This is a symmetric real bilinear form. A complex trigonometric polynomial of degree at most j is a linear combination of $e^{i\ell\theta}$ for $-j \leq \ell \leq j$.

Remark (notation). $S \in \mathbb{S}^3$ is the real symmetric weight in $\langle F, H \rangle_S = \operatorname{Re} \operatorname{tr}(F^* S H)$, with no positive semidefiniteness requirement. In Proposition 4.1: In the matching construction, $S = J_u D J_u^T$, where $u \in \mathbb{R}^4$ and $D \in \mathbb{S}^4$ are arbitrary; neither D nor S must be positive semidefinite. When used as an argument of $\langle F, H \rangle_S$, H is an arbitrary matrix in $\mathbb{C}^{3 \times 2}$. In Theorem 1.1, Proposition 4.1: The notation $H(\Lambda)$ denotes the real symmetric moment matrix with entries $H(\Lambda)_{(i,a),(j,b)} = \Lambda(f_a f_b (L_\theta)_{ij})$ for $1 \leq i, j \leq 20$ and $1 \leq a, b \leq 5$, using $(f_1, \dots, f_5) = (1, \cos \theta, \sin \theta, \cos 2\theta, \sin 2\theta)$.

The finite lift will test L_θ against trigonometric polynomial vectors built from the columns of a matrix $\tilde{C}(\theta)$. A constant Gram matrix for $\tilde{C}$ makes the coefficient of $\mathcal{X}$ in the quadratic contribution of the top blocks independent of θ . Requiring $\langle F_\theta, C \rangle_S = \operatorname{tr}(SQ(Y_\theta, z))$ makes the mixed contribution linear in Y_θ . The next lemma achieves both properties with entries of degree at most two. The matrix S in the pairing may be indefinite.

Remark (notation). For each fixed $S \in \mathbb{S}^3$ and $z \in \mathbb{R}^3$ with $(z_1, z_2) \neq (0, 0)$, $\tilde{C}(\theta)$ is the realification of a chosen matching function $C(\theta)$, with $\theta \in \mathbb{R}$; the choice may depend on S and z . In Proposition 4.1: The chosen matrix is obtained with $S = J_u D J_u^T$, for fixed $u \in \mathbb{R}^4$, $D \in \mathbb{S}^4$, and $z \in \mathbb{R}^3$ satisfying $(z_1, z_2) \neq (0, 0)$.

Lemma 3.1 (Degree-two matching). *Let $S \in \mathbb{S}^3$ and let $z \in \mathbb{R}^3$ satisfy $(z_1, z_2) \neq (0, 0)$. There is a matrix $C(\theta) \in \mathbb{C}^{3 \times 2}$ whose entries are trigonometric polynomials of degree at most two such that, for all $\theta, \eta, \xi \in \mathbb{R}$,*

$$\begin{aligned} \tilde{C}(\theta) \tilde{C}(\theta)^T &= Q(z), \\ \langle F_\theta(\eta, \xi), C(\theta) \rangle_S &= \operatorname{tr}(SQ(Y_\theta(\eta, \xi), z)). \end{aligned} \tag{12}$$

Remark. The linear combination is finite, with coefficients in $\mathbb{C}$ independent of θ and indices $\ell \in \mathbb{Z}$ satisfying $-j \leq \ell \leq j$.

Remark (notation). C is a chosen function of $\theta \in \mathbb{R}$ for each fixed pair $(S, z) \in \mathbb{S}^3 \times \mathbb{R}^3$ satisfying $(z_1, z_2) \neq (0, 0)$; its choice may depend on both S and z . In Proposition 4.1: The matching data are $S = J_u D J_u^T$, with $u \in \mathbb{R}^4$, $D \in \mathbb{S}^4$, and $(z_1, z_2) \neq (0, 0)$ fixed in the argument.

Proof. Write $z = \tau d(\alpha) + \rho k$, where $\tau > 0$, and set

$$w_0 = e^{i\alpha}, \quad R_0 = \tau U(\alpha) + \rho V_0, \quad N = S R_0.$$

By Lemma 2.1, $\tilde{R}_0 \tilde{R}_0^T = Q(z)$. We will set $C(\theta) = R_0 M(\theta)$ with $M(\theta)$ unitary, so that this Gram matrix remains fixed. We choose the rotation to make $\langle V_0, C \rangle_S$ constant and $\langle U(\theta), C(\theta) \rangle_S$ a linear combination of $\cos \theta$ and $\sin \theta$. After those frequency bounds are established, values and a derivative at α will identify the pairings.

Remark (notation). The symbol N has distinct local meanings determined by its scope. In Lemma 3.1: $N = S R_0 \in \mathbb{C}^{3 \times 2}$ is independent of θ . In Theorem 1.1: N denotes the size of the real symmetric

matrix pencil; the construction gives $N = 100$. $M(\theta) = \begin{pmatrix} a(\theta) & b(\theta) \\ -b(\theta) & a(\theta) \end{pmatrix}$ uses the prescribed rotation coefficients, is unitary for every real θ , and satisfies $M(\alpha) = I_2$. $w_0 = e^{i\alpha}$ is a unit complex scalar fixed as θ varies. In Lemma 3.1: The angle α is chosen from $z = \tau d(\alpha) + \rho k$ with $\tau > 0$; equivalently, $w_0 = (z_1 + iz_2)/\sqrt{z_1^2 + z_2^2}$ for $(z_1, z_2) \neq (0, 0)$.

Choosing the unitary rotation. Write N_j for the j th row of N and introduce the two-component row

$$v = N_3 - iN_2.$$

Subscripts $+$ and $-$ denote the first and second components of a row. Set

$$g = v_+ + \overline{v_-}, \quad h = v_+ - \overline{v_-}, \quad \kappa(\theta) = (w_0/w)^2, \quad \delta = |g|^2 + |h|^2.$$

We seek a matrix of the form

$$M(\theta) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

Writing $v^M = vM$, we impose the two conditions

$$v_+^M + \overline{v_-^M} = g, \quad v_+^M - \overline{v_-^M} = h\kappa. \quad (13)$$

The sum on the left controls the pairing with V_0 , while the difference multiplies w^3 in the pairing with U . These conditions therefore keep the first pairing constant and lower the cubic term of the second to frequency one, since $w^3\kappa = w_0^2w$. The expansion below will also bound its remaining frequencies.

Remark (notation). b denotes either the complex rotation coefficient or the finite basis index, according to its local displayed construction. In Lemma 3.1: $b = b(\theta) \in \mathbb{C}$ satisfies $\bar{b} = gh(1 - \kappa)/\delta$ for $\delta > 0$, and $b = 0$ for $\delta = 0$, where $\delta = |g|^2 + |h|^2$ and $\kappa(\theta) = (w_0/w)^2$. In Theorem 1.1, Proposition 4.1: In $H(\Lambda)_{(i,a),(j,b)}$, b is an integer basis index with $1 \leq b \leq 5$.

The two left sides of (13) are $ga + \bar{h}\bar{b}$ and $ha - \bar{g}\bar{b}$, respectively. Thus, for $\delta > 0$, we choose $a(\theta)$ and $b(\theta)$ by solving

$$\begin{pmatrix} g & \bar{h} \\ h & -\bar{g} \end{pmatrix} \begin{pmatrix} a \\ \bar{b} \end{pmatrix} = \begin{pmatrix} g \\ h\kappa \end{pmatrix}. \quad (14)$$

The coefficient matrix A satisfies $A^*A = \delta I_2$, so the system has a unique solution. Explicitly,

$$a = \frac{|g|^2 + |h|^2\kappa}{\delta}, \quad \bar{b} = \frac{gh(1 - \kappa)}{\delta}. \quad (15)$$

Since the right side of (14) has squared norm δ , we have $|a|^2 + |b|^2 = 1$. For $\delta = 0$, take $a = 1$ and $b = 0$; then $g = h = 0$ and the same system is satisfied. In both cases M is unitary, $M(\alpha) = I_2$, and $a, \bar{b}$ involve only the frequencies 0 and -2 . Thus $C = R_0M$ has degree at most two and $C(\alpha) = R_0$. Moreover,

$$\tilde{C}\tilde{C}^T = \text{Re}(R_0MM^*R_0^*) = \text{Re}(R_0R_0^*) = Q(z),$$

which proves the first identity in (12).

Reducing the frequencies of the pairings. The formula for V_0 and (13) yield

$$\sqrt{2}\langle V_0, C \rangle_S = \text{Re}(v_+^M + v_-^M) = \text{Re } g,$$

so this pairing is constant.

Set $m = N_3 + iN_2$, $n = N_1$, and write $m^M = mM$, $n^M = nM$. Expanding c and s in powers of w , the formula for U gives

$$\sqrt{2} \langle U(\theta), C(\theta) \rangle_S = \operatorname{Re} \left[\frac{1}{2} w^3 v_+^M + w \left(\frac{1}{2} m_+^M - n_+^M \right) - \frac{1}{2} w^{-3} v_-^M + w^{-1} \left(-\frac{1}{2} m_-^M - n_-^M \right) \right].$$

Inside the real part, we may replace each negative-power term by its complex conjugate. The resulting expression is

$$\operatorname{Re} \left[\frac{w^3}{2} (v_+^M - \overline{v_-^M}) + w \left(\frac{m_+^M}{2} - n_+^M - \frac{\overline{m_-^M}}{2} - \overline{n_-^M} \right) \right]. \quad (16)$$

The first term has frequency one because $w^3 \kappa = w_0^2 w$. In the second term, the expression in parentheses has only frequencies 0 and -2 : for any fixed row r ,

$$(rM)_+ = r_+ a - r_- \bar{b}, \quad \overline{(rM)_-} = \bar{r}_+ \bar{b} + \bar{r}_- a.$$

Consequently (16) is a real linear combination of $\cos \theta$ and $\sin \theta$, as required.

Remark (notation). The symbol m has distinct local meanings determined by its scope. In Lemma 3.1: $m = N_3 + iN_2 \in \mathbb{C}^{1 \times 2}$ is a row independent of θ . In Theorem 1.1: m denotes the number of auxiliary variables; the construction gives $m = 307$.

Identifying the pairings. At $\theta = \alpha$, polarization of Lemma 2.1 within the plane spanned by $d(\alpha)$ and k gives

$$\langle V_0, R_0 \rangle_S = \operatorname{tr}(SQ(k, z)), \quad (17)$$

$$\langle U(\alpha), R_0 \rangle_S = \operatorname{tr}(SQ(d(\alpha), z)). \quad (18)$$

Both sides of the desired identity $\langle V_0, C(\theta) \rangle_S = \operatorname{tr}(SQ(k, z))$ are constant, so (17) proves it for every θ .

For the pairing with U , both $\langle U(\theta), C(\theta) \rangle_S$ and $\operatorname{tr}(SQ(d(\theta), z))$ are linear combinations of $\cos \theta$ and $\sin \theta$. They agree at α by (18); it remains to compare their derivatives there. The fixed Gram matrix of C gives

$$\langle C(\theta), C(\theta) \rangle_S = \operatorname{tr}(SQ(z)), \quad \langle R_0, C'(\alpha) \rangle_S = 0.$$

Constancy of the pairing with V_0 also gives $\langle V_0, C'(\alpha) \rangle_S = 0$. Since $R_0 = \tau U(\alpha) + \rho V_0$ and $\tau > 0$, it follows that

$$\langle U(\alpha), C'(\alpha) \rangle_S = 0. \quad (19)$$

Next differentiate the identity

$$\langle \tau U(\theta) + \rho V_0, \tau U(\theta) + \rho V_0 \rangle_S = \operatorname{tr}(SQ(\tau d(\theta) + \rho k)),$$

which follows from Lemma 2.1. Evaluating at α and dividing by 2τ gives

$$\langle U'(\alpha), R_0 \rangle_S = \operatorname{tr}(SQ(d'(\alpha), z)).$$

Together with (19), this is the required derivative identity. A real linear combination of $\cos \theta$ and $\sin \theta$ is determined by its value and derivative at a single angle. Hence the pairing with U has the desired value for all θ . Linearity in η, ξ completes the proof. $\square$

4 A finite semidefinite lift

We now replace the angular family of pencils by one finite matrix condition. Evaluation on an angular slice will provide a feasible lift for each point of K . Conversely, the functions supplied by Lemma 3.1 will turn positivity of the finite matrix into the inequalities that characterize the slice, including its singular boundary.

4.1 The matrix condition

Let $\mathcal{T}_j$ be the real vector space of trigonometric polynomials of degree at most j , with basis

$$1, \cos \theta, \sin \theta, \dots, \cos(j\theta), \sin(j\theta).$$

Let $\mathcal{E}$ be the space of real functions of $(\theta, \mathcal{X}, \mathcal{Z}, \eta, \xi)$ that are linear homogeneous in the 22 coordinates of $(\mathcal{X}, \mathcal{Z}, \eta, \xi) \in \mathbb{S}^4 \times \mathbb{S}^4 \times \mathbb{R}^2$, with coefficients in $\mathcal{T}_7$. Thus $\dim \mathcal{E} = 22 \cdot 15 = 330$. Our lifting variable is a real linear functional $\Lambda : \mathcal{E} \rightarrow \mathbb{R}$; on vectors and matrices we apply it entrywise. For an output point (X, Z, y) , impose

$$\Lambda(\mathcal{X}) = X, \quad \Lambda(\mathcal{Z}) = Z, \quad \Lambda(Y_\theta(\eta, \xi)) = y. \quad (20)$$

We also require

$$\Lambda(q(\theta)^T L_\theta(\mathcal{X}, \mathcal{Z}, \eta, \xi) q(\theta)) \geq 0 \quad \text{for every } q \in (\mathcal{T}_2)^{20}. \quad (21)$$

The expression to which Λ is applied lies in $\mathcal{E}$: the entries of L_θ have trigonometric degree at most three, so its product with two entries of q has degree at most seven.

Condition (21) is a single finite semidefinite condition, using the finite test-space construction familiar from moment relaxations [Las01, Las09]. Indeed, set $(f_1, \dots, f_5) = (1, \cos \theta, \sin \theta, \cos 2\theta, \sin 2\theta)$ and define $H(\Lambda) \in \mathbb{S}^{100}$ by

$$H(\Lambda)_{(i,a),(j,b)} = \Lambda(f_a f_b (L_\theta)_{ij}), \quad 1 \leq i, j \leq 20, \quad 1 \leq a, b \leq 5. \quad (22)$$

If $q_i = \sum_a t_{i,a} f_a$, the left side of (21) is $t^T H(\Lambda) t$. Hence (21) is equivalent to $H(\Lambda) \succeq 0$. Every entry of this real symmetric matrix is linear homogeneous in the 330 coordinates of Λ .

Remark (notation). $H(\Lambda)$ is indexed by pairs (i, a) with $1 \leq i \leq 20$ and $1 \leq a \leq 5$, and its $((i, a), (j, b))$ entry is $\Lambda(f_a f_b (L_\theta)_{ij})$.

Proposition 4.1. *For $(X, Z, y) \in \mathbb{S}^4 \times \mathbb{S}^4 \times \mathbb{R}^3$, the following are equivalent:*

1. $(X, Z, y) \in K$;
2. there is a real linear functional $\Lambda : \mathcal{E} \rightarrow \mathbb{R}$ satisfying (20) and $H(\Lambda) \succeq 0$.

Proof. Suppose first that $(X, Z, y) \in K$. Choose θ_0, η_0, ξ_0 with $y = Y_{\theta_0}(\eta_0, \xi_0)$, and let Λ be evaluation at $(\theta_0, X, Z, \eta_0, \xi_0)$. The output equations hold, and Lemma 2.2 gives $L_{\theta_0}(X, Z, \eta_0, \xi_0) \succeq 0$, which implies (21). This evaluation functional is a finite feasible witness even when X is singular.

For the converse, suppose that Λ satisfies the stated conditions. Taking constant q supported on one of the four top blocks of L_θ in (21) gives $X \succeq 0$. We next obtain a family of scalar inequalities from which the remaining slice conditions will follow.

Fix $u \in \mathbb{R}^4$, $D \in \mathbb{S}^4$, and $z \in \mathbb{R}^3$ with $(z_1, z_2) \neq (0, 0)$, and apply Lemma 3.1 with $S = J_u D J_u^T \in \mathbb{S}^3$. Neither D nor S is required to be positive semidefinite. Let $b_1(\theta), \dots, b_4(\theta)$ be the columns of

the resulting real matrix $\tilde{C}(\theta)$. Choose the five four-dimensional blocks of q to be

$$q(\theta) = \begin{pmatrix} -DJ_u^T b_1(\theta) \\ \vdots \\ -DJ_u^T b_4(\theta) \\ u \end{pmatrix}.$$

The degree bound in Lemma 3.1 puts this vector in $(\mathcal{T}_2)^{20}$. Write r_j for the columns of $\tilde{F}_\theta(\eta, \xi)$, as in the definition of L_θ . Using $B(r_j)u = J_u^T r_j$, the top blocks and the cross terms of $q^T L_\theta q$ are respectively

$$\begin{aligned} \sum_{j=1}^4 b_j^T J_u D \mathcal{X} D J_u^T b_j &= \text{tr}(J_u D \mathcal{X} D J_u^T Q(z)), \\ -2 \sum_{j=1}^4 b_j^T J_u D J_u^T r_j &= -2 \langle F_\theta(\eta, \xi), C(\theta) \rangle_S \\ &= -2 \text{tr}(J_u D J_u^T Q(Y_\theta(\eta, \xi), z)). \end{aligned}$$

These identities hold before applying Λ . The first identity removes the angular dependence from the coefficient of $\mathcal{X}$; the last expression is linear in $Y_\theta(\eta, \xi)$. Consequently, applying (20) and (21) gives

$$u^T Z u + \text{tr}(J_u D \mathcal{X} D J_u^T Q(z)) - 2 \text{tr}(J_u D J_u^T Q(y, z)) \geq 0. \quad (23)$$

For fixed u, D , the left side is a polynomial in z , so the inequality extends by continuity to every $z \in \mathbb{R}^3$. Only this scalar inequality is extended; no continuity of the functions C is needed. Thus (23) holds for all $u \in \mathbb{R}^4$, $D \in \mathbb{S}^4$, and $z \in \mathbb{R}^3$.

Remark (notation). $b_j(\theta)$, $1 \leq j \leq 4$, are the columns of $\tilde{C}(\theta)$ and are real trigonometric polynomials of degree at most two with values in $\mathbb{R}^3$. In Proposition 4.1: For each fixed $u \in \mathbb{R}^4$, $D \in \mathbb{S}^4$, and $z \in \mathbb{R}^3$ with $(z_1, z_2) \neq (0, 0)$, the columns are chosen from a matching matrix for $S = J_u D J_u^T$ and z ; this choice may depend on u, D, z , and neither D nor S is required to be positive semidefinite.

If X were positive definite, setting $z = y$ and $D = X^{-1}$ in (23) would give $Z \succeq \Phi_y(X^{-1})$. For singular X , the same family of tests also supplies the necessary range condition. Fix slice parameters $y = Y_{\theta_0}(\eta_0, \xi_0)$ and, for the rest of the proof, let r_j denote the columns of $\tilde{F}_{\theta_0}(\eta_0, \xi_0)$ and set $B_j = B(r_j)$. For $v \in \ker X$, take $D = t v v^T$ and $z = y$ in (23), where $t \in \mathbb{R}$ is arbitrary. Since $D X D = 0$, this gives

$$u^T Z u - 2t (J_u v)^T Q(y) (J_u v) \geq 0 \quad (t \in \mathbb{R}).$$

Its coefficient of t must vanish. The Gram factorization of $Q(y)$ therefore yields

$$0 = (J_u v)^T Q(y) (J_u v) = \sum_{j=1}^4 (v^T B_j u)^2.$$

As u and $v \in \ker X$ are arbitrary, it follows that

$$\text{range } B_j \subseteq (\ker X)^\perp = \text{range } X \quad (1 \leq j \leq 4). \quad (24)$$

Remark (notation). B_j denotes a real matrix in a locally specified family. In Theorem 1.1: $B_j \in \mathbb{S}^{100}$, $1 \leq j \leq 307$, are fixed auxiliary coefficient matrices of the homogeneous pencil. In Proposition 4.1:

After fixing $y = Y_{\theta_0}(\eta_0, \xi_0)$, $B_j = B(r_j)$ for $1 \leq j \leq 4$, where r_j are the columns of $\tilde{F}_{\theta_0}(\eta_0, \xi_0)$ and $B(r) = \begin{pmatrix} 0 & r^T \\ -r & 0_{3 \times 3} \end{pmatrix} \in \mathbb{R}^{4 \times 4}$.

Let $X^\dagger$ be the symmetric matrix that inverts the positive eigenvalues of X and is zero on $\ker X$. In particular, $X^\dagger X X^\dagger = X^\dagger$. Taking $D = X^\dagger$ and $z = y$ in (23), and using (8) and (7), gives

$$Z \succeq \Phi_y(X^\dagger) = \sum_{j=1}^4 B_j^T X^\dagger B_j. \quad (25)$$

We now verify positivity of the slice pencil directly by completing a square using (24) and (25). For $a_1, \dots, a_4, u \in \mathbb{R}^4$, its quadratic form equals

$$\begin{aligned} & \sum_{j=1}^4 a_j^T X a_j + 2 \sum_{j=1}^4 a_j^T B_j u + u^T Z u \\ &= \sum_{j=1}^4 (a_j + X^\dagger B_j u)^T X (a_j + X^\dagger B_j u) + u^T \left(Z - \sum_{j=1}^4 B_j^T X^\dagger B_j \right) u \geq 0. \end{aligned}$$

Here the cross terms agree because $XX^\dagger B_j = B_j$ by (24). Thus $L_{\theta_0}(X, Z, \eta_0, \xi_0) \succeq 0$, and Lemma 2.2 gives $(X, Z, y) \in K$. $\square$

The converse has used an arbitrary feasible functional Λ . In particular, the proof requires no representation of Λ by a measure and no closure operation on the projected feasible set.

4.2 The size of the lift

Proof of Theorem 1.1. Choose the displayed trigonometric basis of $\mathcal{T}_7$ and the upper-triangular coordinate functions of $\mathcal{X}$ and $\mathcal{Z}$, together with η, ξ , to specify the 330 coordinates of Λ . The output equations (20) prescribe precisely the following 23 distinct coordinates:

$$\begin{aligned} & \Lambda(\mathcal{X}_{ij}), \quad \Lambda(\mathcal{Z}_{ij}) \quad (1 \leq i \leq j \leq 4), \\ & \Lambda(\eta \cos \theta), \quad \Lambda(\eta \sin \theta), \quad \Lambda(\xi). \end{aligned}$$

They are independent members of the chosen coordinate system on $\mathcal{E}^*$. Substitute the corresponding entries of X, Z, y for these coordinates in (22), and use the remaining $330 - 23 = 307$ coordinates as auxiliary variables. The result is a fixed real symmetric 100×100 matrix pencil, homogeneous in the output and auxiliary variables. Proposition 4.1 proves that its projection is exactly K . Hence the requested representation has $N = 100$, $m = 307$, and zero constant term. $\square$

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