

Near Inclusions of von Neumann Algebras Without Small Spatial Embeddings

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Abstract

We give a negative answer to the unrestricted small-spatial-embedding problem for one-sided near inclusions of von Neumann algebras. On separable complex Hilbert spaces, we construct pairs of unital von Neumann algebras with common identity whose one-sided gaps tend to zero. Spatial embeddings of the source into the target exist, but every implementing unitary stays a fixed positive distance from the identity. The obstruction therefore concerns small implementing unitaries, rather than the existence of spatial embeddings.

1 Introduction

Let M and N be von Neumann algebras acting on a complex Hilbert space H , both with identity I_H . Their one-sided near-inclusion gap is

$$\gamma(M, N) = \sup_{\substack{x \in M \\ \|x\| \leq 1}} \inf_{y \in N} \|x - y\|. \quad (1.1)$$

The approximating element y is not required to be contractive. A small value of $\gamma(M, N)$ says that every contraction in M is close to some element of N ; it does not require these choices of approximants to preserve algebraic operations. If a unitary u satisfies $uMu^* \subseteq N$, then $\gamma(M, N) \leq 2\|u - I_H\|$. The converse question asks whether a small near inclusion can always be obtained in this way.

Remark (notation). H is the common complex Hilbert space on which M and N act. In Theorem 1.1, Proposition 3.1: In the algebra construction, $H = S \otimes L$, where $G = F \times F$, $S = \ell^2(G)$ and $L = K \otimes K \otimes D$. $M \subseteq \mathcal{B}(H)$ is a von Neumann algebra with identity I_H , serving as the source argument of $\gamma(M, N)$. In Theorem 1.1, Proposition 3.1: In the algebra construction, $M = W(\lambda(G)'' \otimes I_L)W^*$; here $\lambda(G)'' \otimes I_L$ consists of all $X \otimes I_L$ with $X \in \lambda(G)''$. $N \subseteq \mathcal{B}(H)$ is a von Neumann algebra with identity I_H , serving as the target argument of $\gamma(M, N)$. In Theorem 1.1, Proposition 3.1: In the algebra construction, $N = (\rho(G) \otimes I_L)'$, the commutant of the amplified right regular representation.

The underlying perturbation program began with Kadison and Kastler [8]. They measured the distance between operator algebras by the Hausdorff distance between their unit balls in the operator norm, now called the Kadison–Kastler distance, and proposed that sufficiently close von Neumann algebras should be related by a unitary close to the identity. Christensen, Johnson, and Raeburn–Taylor established the amenable case in the 1970s [2, 7, 10]. Christensen subsequently introduced near inclusions and proved a spatial embedding theorem for an amenable source

algebra and an arbitrary target [3, Definition 2.1 and Theorem 4.3]. In particular, sufficiently small near inclusions in this setting admit implementing unitaries whose distance from the identity is bounded linearly by the near-inclusion tolerance.

We use the following formulation of the unrestricted one-sided question posed by Cameron, Christensen, Sinclair, Smith, White, and Wiggins in the paragraph preceding [1, Theorem 2]:

For every $\varepsilon > 0$, is there a $\delta > 0$ such that, for every complex Hilbert space H and every pair of von Neumann algebras $M, N \subseteq \mathcal{B}(H)$ with common identity I_H , $\gamma(M, N) < \delta$ implies the existence of a unitary $u \in \mathcal{B}(H)$ satisfying

$$uMu^* \subseteq N \quad \text{and} \quad \|u - I_H\| < \varepsilon?$$

Here δ must be independent of the algebras and their representations. We answer this question negatively.

Theorem 1.1. *There exist $\varepsilon_0 > 0$, separable complex Hilbert spaces H_n , and von Neumann algebras $M_n, N_n \subseteq \mathcal{B}(H_n)$ with common identity I_{H_n} such that*

$$\gamma(M_n, N_n) \longrightarrow 0,$$

but, for all sufficiently large n , every unitary $u \in \mathcal{B}(H_n)$ satisfying $uM_nu^* \subseteq N_n$ obeys

$$\|u - I_{H_n}\| \geq \varepsilon_0.$$

Remark (notation). The constant $\varepsilon_0 > 0$ is independent of n and of the implementing unitary u ; its lower-bound property holds for all sufficiently large n . (H_n) is a sequence of complex Hilbert spaces. In Lemma 4.2: H_n is arbitrary and need not be separable. In Theorem 1.1: H_n is separable and is realized as $K \otimes K \otimes K \otimes K \otimes \mathbb{C}^n$ under $S = K \otimes K$. The sequence M_n is indexed by integers $n \geq 3$. The sequence N_n is indexed by integers $n \geq 3$.

In these examples a unitary embedding of M_n into N_n does exist; the obstruction concerns its distance from the identity. The theorem therefore separates normwise approximation from small spatial implementation even on separable Hilbert spaces. Its hypothesis is one-sided. The construction supplies no reverse near inclusion, and hence does not give a counterexample to the two-sided Kadison–Kastler conjecture. The positive results for nonamenable factors in [1], which concern two-sided perturbations under additional structural hypotheses, address a different assertion.

The proof transfers a classical finite-dimensional obstruction into the near-inclusion problem. Voiculescu’s clock and shift matrices $U_n, V_n \in \mathcal{B}(\mathbb{C}^n)$ satisfy $\|U_n V_n - V_n U_n\| \rightarrow 0$, but cannot be approximated in norm by commuting unitary pairs [11]. Exel and Loring gave an elementary winding-number proof of this phenomenon [5]; we use its trace-logarithm form [4, Lemma 3.1]. We recall the short obstruction argument in Section 4.

Remark (notation). U_n is defined for every integer $n \geq 3$ by $U_n e_j = \omega_n^j e_j$ for $0 \leq j < n$. V_n is defined for every integer $n \geq 3$ by $V_n e_j = e_{j+1 \bmod n}$ for $0 \leq j < n$.

To place these matrices in operator algebras, we use the free group F on countably many generators. The starting representation is the real-parameter Pytlik–Szwarc representation [9, §2.3, Equation (5)], whose cyclic coefficient is Haagerup’s positive-definite function $g \mapsto t^{|g|}$, $0 < t < 1$, where $|g|$ is reduced word length [6, Lemma 1.2]. Section 2 gives a self-contained tree realization at $t = 1/2$ and adds a circle parameter z . The norms of its Fourier coefficients are bounded, uniformly over the group, by a sequence with summable first moment. Along successive

distinct free generators, the values of the representations converge in the weak operator topology to tz times the projection onto the identity vector in $\ell^2(F)$. Substituting U_n and V_n for the circle parameter therefore produces two unitary representations of F whose weak limits retain the original matrices.

Two further steps connect this representation family to the theorem. In Section 3, a diagonal unitary built from the two representations conjugates a copy of the group von Neumann algebra of $F \times F$. The target N_n is the commutant of an amplified right regular representation of this product group. Reordering the Fourier expansions gives an approximant in N_n and an error bounded by a constant times $\|U_n V_n - V_n U_n\|$. Crucially, this estimate holds on the entire unit ball of the source algebra. In Section 4, a hypothetical sequence of spatial embeddings tending to the identity places the conjugated left regular representation in N_n , where it commutes with the right regular representation. Their pointwise product is then a representation of $F \times F$, so the von Neumann algebras generated by its two factors commute. Weak operator limits in these algebras recover operators close to copies of U_n and V_n supported on projections. Perturbing those projections, compressing to a common finite-dimensional corner, and taking polar parts would yield commuting unitary matrices close to U_n, V_n , contradicting the winding-number obstruction.

2 A circle family with uniform Fourier bounds

We first construct representations in which a scalar circle parameter can later be replaced by a unitary matrix. Two properties are needed: Fourier coefficients with a summable first moment, uniformly over the group, and a weak operator limit that retains the parameter.

Let F be the free group on generators s_j , $j \geq 1$, with identity e . Put $K = \ell^2(F)$, write $(\delta_x)_{x \in F}$ for its standard orthonormal basis, and let p_e be the projection onto $\mathbb{C}\delta_e$. We use the left regular representation $l(g)\delta_x = \delta_{gx}$. Throughout the construction,

$$t = \frac{1}{2}, \quad r = (1 - t^2)^{1/2}, \quad \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

Remark (notation). δ_x denotes the standard basis vector indexed by x in the ambient group Hilbert space: $K = \ell^2(F)$ when $x \in F$, and $S = \ell^2(G)$ when $x \in G$.

Proposition 2.1. *There are unitary representations $\pi_z : F \rightarrow \mathcal{B}(K)$, indexed by $z \in \mathbb{T}$, and operators $A_k(g) \in \mathcal{B}(K)$, indexed by $k \in \mathbb{Z}$ and $g \in F$, such that*

$$\pi_z(g) = \sum_{k \in \mathbb{Z}} z^k A_k(g), \quad \sup_{g \in F} \|A_k(g)\| \leq c_k, \quad c_k = \mathbf{1}_{\{k=0\}} + \frac{2t^{|k|}}{1-t}, \quad (2.1)$$

$$\pi_z(s_j) \xrightarrow{j \rightarrow \infty} \text{WOT} tzp_e. \quad (2.2)$$

The series in (2.1) converges absolutely in operator norm, uniformly in g and z , and

$$S_0 := \sum_{k \in \mathbb{Z}} c_k < \infty, \quad S_1 := \sum_{k \in \mathbb{Z}} |k|c_k < \infty. \quad (2.3)$$

Remark (notation). $\pi_z : F \rightarrow \mathcal{B}(K)$ is a unitary representation for each $z \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The operators $A_k(g) \in \mathcal{B}(K)$ are indexed by $k \in \mathbb{Z}$ and $g \in F$ and are the Laurent coefficients in $\pi_z(g) = \sum_{k \in \mathbb{Z}} z^k A_k(g)$ for $z \in \mathbb{T}$. The sequence is indexed by all $k \in \mathbb{Z}$ and is defined by $c_k = \mathbf{1}_{\{k=0\}} + 2t^{|k|}/(1-t)$, with $t = 1/2$.

Proof. The underlying representation is the real-parameter Pytlik–Szwarc representation [9, Section 2.3, Equation (5)]; its cyclic coefficient is the positive-definite function $g \mapsto t^{|g|}$ of Haagerup [6, Lemma 1.2]. We give the tree construction before introducing the circle parameter.

Use the Cayley tree whose edges join x to $xs_j^{\pm 1}$, rooted at e , and write $p(x)$ for the parent of $x \neq e$. Define

$$\xi_e = \delta_e, \quad \xi_x = t\xi_{p(x)} + r\delta_x \quad (x \neq e). \quad (2.4)$$

All these vectors have norm one. If v is the last common vertex of the paths from e to x and y , the two tails beyond v are orthogonal to each other and to ξ_v . Thus

$$\langle \xi_x, \xi_y \rangle = t^{d(x,v)+d(y,v)} = t^{d(x,y)},$$

where d is the tree distance. Their linear span contains every δ_x , by (2.4). Left multiplication preserves d , so the rule

$$\pi(g)\xi_x = \xi_{gx}$$

extends to a unitary on K . These unitaries form a representation.

The difference from the regular representation is confined to a single finite path. To see this precisely, put $T_g = \pi(g)l(g)^*$. For $y \neq g$, (2.4) gives

$$T_g\delta_y = \frac{\xi_y - t\xi_{gp(g^{-1}y)}}{r}, \quad T_g\delta_g = \xi_g. \quad (2.5)$$

The vertex $gp(g^{-1}y)$ is the first step from y toward g . Let $e = h_0, h_1, \dots, h_d = g$ be the path from e to g . For y outside this path, the first steps toward e and g coincide; hence $T_g\delta_y = \delta_y$. On the path,

$$T_g\delta_{h_b} = r\xi_{h_b} - t\delta_{h_{b+1}} \quad (0 \leq b < d), \quad T_g\delta_{h_d} = \xi_{h_d}. \quad (2.6)$$

In particular, the path subspace and its orthogonal complement reduce T_g . Expanding the vectors along the path gives

$$\xi_{h_b} = t^b\delta_{h_0} + r \sum_{a=1}^b t^{b-a}\delta_{h_a}.$$

By (2.6), the subdiagonal entries are $-t$; every other nonzero entry in row a , column b is $t^{|a-b|}$ times 1, r , or r^2 . Thus the matrix entries on this subspace satisfy

$$|(T_g)_{a,b}| \leq t^{|a-b|} \quad (0 \leq a, b \leq d). \quad (2.7)$$

This also includes $g = e$, when $T_e = I$.

Let $q : F \rightarrow \mathbb{Z}$ be the homomorphism with $q(s_j) = 1$ for every j , and define $D_z\delta_x = z^{-q(x)}\delta_x$. The desired representation is

$$\pi_z(g) = z^{q(g)}D_z\pi(g)D_z^* = (D_zT_gD_z^*)l(g). \quad (2.8)$$

The first expression proves that it is a unitary representation. The second follows from $D_zl(g)D_z^* = z^{-q(g)}l(g)$ and supplies its Fourier bounds.

Remark (notation). D is the arbitrary nonzero finite-dimensional complex Hilbert space carrying the input unitaries U, V . In Theorem 1.1: For each integer $n \geq 3$, the construction is applied with $D = \mathbb{C}^n$.

On the path subspace, the entry in row a , column b acquires the Laurent factor $z^{q(h_b)-q(h_a)}$. Each edge changes q by 1 or -1 , so an entry of exponent k can occur only when $|a-b| \geq |k|$.

For each row or column there are at most two entries at any given distance m . By (2.7), the absolute row and column sums of the coefficient of z^k are therefore at most

$$2 \sum_{m \geq |k|} t^m = \frac{2t^{|k|}}{1-t}.$$

Outside the path the operator is the identity, which contributes only to the coefficient at $k = 0$. The row and column sum bound for operator norm, followed by multiplication by $l(g)$, proves (2.1). For each fixed g this is a finite Laurent expansion; the common bounds c_k give the asserted uniform absolute convergence and (2.3).

Finally, (2.6) and (2.8) give

$$\pi_z(s_j)\delta_e = tz\delta_e + r\delta_{s_j}.$$

For any fixed $x \neq e$, all sufficiently large j satisfy $s_jx \notin \{e, s_j\}$, and then $\pi_z(s_j)\delta_x = \delta_{s_jx}$. These basis-vector calculations, together with the uniform norm bound $\|\pi_z(s_j)\| = 1$, prove (2.2). $\square$

3 From a unitary pair to a near inclusion

We now use Proposition 2.1 to encode two unitaries U, V on a nonzero finite-dimensional Hilbert space D . The resulting von Neumann algebras will have one-sided gap bounded by a constant times $\|UV - VU\|$.

3.1 Matrix substitution and the algebras

On $L = K \otimes K \otimes D$, define

$$a(g) = \sum_k A_k(g) \otimes I_K \otimes U^k, \quad b(g) = \sum_k I_K \otimes A_k(g) \otimes V^k \quad (g \in F). \quad (3.1)$$

The series converge absolutely in norm. A spectral decomposition $U = \sum_z zP_z$ rewrites the first expression as $\sum_z \pi_z(g) \otimes I_K \otimes P_z$, so a is a unitary representation. Diagonalizing V gives the same conclusion for b . There is no requirement that their ranges commute. The weak limits from Proposition 2.1 become

$$a(s_j) \xrightarrow{\text{WOT}} tp_e \otimes I_K \otimes U, \quad b(s_j) \xrightarrow{\text{WOT}} tI_K \otimes p_e \otimes V. \quad (3.2)$$

Only finite spectral sums are involved, so these limits follow directly from (2.2).

Put $G = F \times F$, $S = \ell^2(G)$, and $H = S \otimes L$. Our conventions for the commuting left and right regular representations on S are

$$\lambda(g)\delta_x = \delta_{gx}, \quad \rho(g)\delta_x = \delta_{xg^{-1}}.$$

For $x = (x_1, x_2) \in G$, put $f(x) = a(x_1)b(x_2)$ and define the diagonal unitary W by

$$W(\delta_x \otimes v) = \delta_x \otimes f(x)v.$$

Although a and b are representations, f need not be one. Define

$$M = W(\lambda(G)'' \otimes I_L)W^*, \quad N = (\rho(G) \otimes I_L)'. \quad (3.3)$$

Primes denote commutants. Here $\lambda(G)'' \otimes I_L$ means the algebra of all $X \otimes I_L$ with $X \in \lambda(G)''$; it is weak operator closed, as is seen by taking matrix blocks relative to an orthonormal basis of L . Thus M, N are von Neumann algebras with common identity I_H . Notice also that $W^*MW \subset N$, so these examples always admit a spatial embedding.

The exactly commuting case explains the choice of W . If U and V commute, then so do the ranges of a and b , and f is a representation of G . In that case

$$W(\lambda(g) \otimes I_L)W^* = \lambda(g) \otimes f(g) \in N \quad (g \in G),$$

so $M \subset N$. When U and V only nearly commute, we must control the error on the entire unit ball of M ; estimates for the group elements alone would not suffice.

3.2 A norm estimate on the whole algebra

Proposition 3.1. *For the algebras in (3.3),*

$$\gamma(M, N) \leq C\|UV - VU\|, \quad C := \sum_{k,m,p,h \in \mathbb{Z}} c_k c_m c_p c_h |m - p| |h| < \infty. \quad (3.4)$$

Moreover, for every $g \in G$,

$$\|W(\lambda(g) \otimes I_L)W^* - \lambda(g) \otimes f(g)\| \leq C\|UV - VU\|. \quad (3.5)$$

The constant is independent of D, U, V .

Remark. For every $g \in F$, both sums defining $a(g)$ and $b(g)$ run over $k \in \mathbb{Z}$ and converge absolutely in operator norm.

Proof. Write $\nu = \|UV - VU\|$. Fix $X \in \lambda(G)''$ and denote its scalar matrix entries by $X_{x,y}$, for $x, y \in G$. Since X commutes with $\rho(G)$, these entries are invariant under common right translation:

$$X_{xq,yq} = X_{x,y} \quad (x, y, q \in G). \quad (3.6)$$

The operator $Z = W(X \otimes I_L)W^*$ has $\mathcal{B}(L)$ -valued blocks

$$Z_{x,y} = X_{x,y} a(x_1) b(x_2) b(y_2)^* a(y_1)^*.$$

We seek an approximant Y with the reordered blocks

$$Y_{x,y} = X_{x,y} a(x_1) a(y_1)^* b(x_2) b(y_2)^* = X_{x,y} f(xy^{-1}). \quad (3.7)$$

The last equality uses the representation laws for a and b separately. Since $(xq)(yq)^{-1} = xy^{-1}$, these blocks are right-translation invariant by (3.6). Thus, once boundedness is established, Y commutes with $\rho(G) \otimes I_L$ and belongs to N .

The Fourier bounds establish boundedness and control the reordering in operator norm. On $S \otimes K \otimes K$, let $D_{k,m}$ be the diagonal operator whose block at $x \in G$ is $A_k(x_1) \otimes A_m(x_2)$. Then $\|D_{k,m}\| \leq c_k c_m$. Set

$$B_{k,m,p,h}(X) = D_{k,m}(X \otimes I_K \otimes I_K) D_{h,p}^*.$$

The A_k coefficients of a and b act on separate copies of K ; only their matrix factors U^k, V^m act on the same space. In the decomposition $H = (S \otimes K \otimes K) \otimes D$, expansion of the four factors therefore gives

$$Z = \sum_{k,m,p,h} B_{k,m,p,h}(X) \otimes U^k V^{m-p} U^{-h}, \quad (3.8)$$

$$Y = \sum_{k,m,p,h} B_{k,m,p,h}(X) \otimes U^{k-h} V^{m-p}. \quad (3.9)$$

Both sums converge absolutely because

$$\|B_{k,m,p,h}(X)\| \leq c_k c_m c_p c_h \|X\|, \quad \sum_{k,m,p,h} c_k c_m c_p c_h = S_0^4 < \infty.$$

Their blocks are precisely the expressions above, which also justifies the definition of Y .

For integers a, b , telescoping gives

$$\|U^a V^b - V^b U^a\| \leq |a| |b| \nu. \quad (3.10)$$

For positive powers, expand a commutator one factor at a time; negative powers follow from $[U^{-1}, V] = -U^{-1}[U, V]U^{-1}$ and its analogue for V^{-1} , where $[A, B] = AB - BA$. Consequently the two matrix factors in (3.8) and (3.9) differ by at most $|m - p| |h| \nu$. Hence

$$\|Z - Y\| \leq C \nu \|X\|, \quad C \leq 2S_0^2 S_1^2 < \infty. \quad (3.11)$$

Since $Y \in N$ and $\|Z\| = \|X\|$, this proves (3.4); no bound of 1 on $\|Y\|$ is needed.

If $X = \lambda(g)$, its nonzero entries have $x = gy$. The representation laws for a and b turn (3.7) into $Y = \lambda(g) \otimes a(g_1)b(g_2) = \lambda(g) \otimes f(g)$. Thus (3.5) is the same estimate for this choice of X . $\square$

4 Recovering the matrix obstruction

The preceding construction turns a small commutator into a small one-sided gap. We now show that a spatial embedding close to the identity would turn the original matrix pair into a nearby commuting pair. Choosing the clock and shift matrices will make this impossible.

4.1 The winding obstruction

The matrices below are Voiculescu's almost commuting unitary pair [11]. We include the logarithmic winding argument, in the form developed in [5, 4].

Lemma 4.1. *For $n \geq 3$, let U_n, V_n be the unitaries on $\mathbb{C}^n$ given by*

$$U_n e_j = \omega_n^j e_j, \quad V_n e_j = e_{j+1 \bmod n}, \quad \omega_n = e^{2\pi i/n} \quad (0 \leq j < n). \quad (4.1)$$

Then $\nu_n := \|U_n V_n - V_n U_n\| = |\omega_n - 1| \rightarrow 0$. There is no infinite subsequence along which commuting unitaries $\hat{U}_n, \hat{V}_n$ on $\mathbb{C}^n$ satisfy

$$\|\hat{U}_n - U_n\| + \|\hat{V}_n - V_n\| \rightarrow 0.$$

Remark (notation). For each integer $n \geq 3$, $\omega_n = e^{2\pi i/n} \in \mathbb{T}$. For each integer $n \geq 3$, $\nu_n = \|U_n V_n - V_n U_n\| = |\omega_n - 1|$ for the clock-shift pair on $\mathbb{C}^n$. $\widehat{U}_n$ denotes the first member of a hypothetical pair of commuting unitaries $\widehat{U}_n, \widehat{V}_n$ on $\mathbb{C}^n$, indexed along an infinite subsequence of integers $n \geq 3$, satisfying $\|\widehat{U}_n - U_n\| + \|\widehat{V}_n - V_n\| \rightarrow 0$ along that subsequence. $\widehat{V}_n$ denotes a hypothetical unitary on $\mathbb{C}^n$ indexed by an infinite subsequence of integers $n \geq 3$, commuting with $\widehat{U}_n$, for which $\|\widehat{U}_n - U_n\| + \|\widehat{V}_n - V_n\| \rightarrow 0$ along that subsequence.

Proof. The identity $U_n V_n = \omega_n V_n U_n$ proves the first assertion. For any unitary matrices U, V whose commutator UVU^*V^* has no eigenvalue -1 , define

$$\kappa(U, V) = \frac{1}{2\pi i} \operatorname{Tr} \log(UVU^*V^*),$$

using the principal logarithm and the unnormalized trace. The determinant identity

$$\exp(\operatorname{Tr} \log(UVU^*V^*)) = \det(UVU^*V^*) = 1$$

shows that $\kappa(U, V)$ is an integer. It is continuous, and hence constant, along any path on which the logarithm remains defined. For the pair in (4.1), its value is 1, since $U_n V_n U_n^* V_n^* = \omega_n I$ and $0 < 2\pi/n < \pi$. For a commuting pair its value is 0.

Suppose the asserted approximants existed along a subsequence. For all sufficiently large n on that subsequence, write $U_n^* \widehat{U}_n = e^{iA_n}$ and $V_n^* \widehat{V}_n = e^{iB_n}$, with A_n, B_n self-adjoint and spectra in $(-\pi, \pi)$. The paths

$$U_n(s) = U_n e^{isA_n}, \quad V_n(s) = V_n e^{isB_n} \quad (0 \leq s \leq 1)$$

stay within the respective endpoint distances from U_n, V_n . Indeed, for $|\theta| < \pi$ and $0 \leq s \leq 1$, $|e^{is\theta} - 1| \leq |e^{i\theta} - 1|$. A four-factor telescoping estimate gives, uniformly in s ,

$$\|U_n(s)V_n(s)U_n(s)^*V_n(s)^* - I\| \leq \nu_n + 2\|\widehat{U}_n - U_n\| + 2\|\widehat{V}_n - V_n\| \rightarrow 0.$$

The principal logarithm is therefore defined along the whole path for all sufficiently large n . Its integer changes from 1 to 0, a contradiction. $\square$

4.2 A finite-dimensional corner

Compression of commuting operators to a prescribed subspace need not preserve commutation. The next lemma avoids this difficulty by replacing each support projection by a nearby projection in its corresponding algebra and then using their common range.

Lemma 4.2. *For each n , let E_{1n}, E_{2n} be commuting projections on a complex Hilbert space H_n , with $D_n = E_n H_n$ nonzero and finite-dimensional, where $E_n = E_{1n} E_{2n}$. Let $\tilde{R}_{1n}, \tilde{R}_{2n}$ be unitaries commuting with both projections, and write $R_{in} = \tilde{R}_{in}|_{D_n}$. Suppose $\mathcal{A}_{1n}, \mathcal{A}_{2n} \subset \mathcal{B}(H_n)$ are commuting unital von Neumann algebras and there are $T_{in} \in \mathcal{A}_{in}$ such that*

$$\|T_{in} - E_{in} \tilde{R}_{in}\| \rightarrow 0 \quad (i = 1, 2). \quad (4.2)$$

Then, for all sufficiently large n , there are commuting unitaries $\widehat{R}_{1n}, \widehat{R}_{2n}$ on D_n with

$$\|\widehat{R}_{in} - R_{in}\| \rightarrow 0 \quad (i = 1, 2).$$

Remark (notation). E_{1n} is an orthogonal projection on H_n commuting with E_{2n} . In Theorem 1.1: $E_{1n} = p_e \otimes I_K \otimes p_e \otimes I_K \otimes I_{\mathbb{C}^n}$ on $H_n = K \otimes K \otimes K \otimes K \otimes \mathbb{C}^n$. E_{2n} is an orthogonal projection on H_n commuting with E_{1n} . In Theorem 1.1: $E_{2n} = I_K \otimes p_e \otimes I_K \otimes p_e \otimes I_{\mathbb{C}^n}$ on $H_n = K \otimes K \otimes K \otimes K \otimes \mathbb{C}^n$. $E_n = E_{1n}E_{2n}$ is the orthogonal projection onto the intersection of the ranges of the commuting projections E_{1n} and E_{2n} . In Theorem 1.1: $E_n = p_e \otimes p_e \otimes p_e \otimes p_e \otimes I_{\mathbb{C}^n}$ on $H_n = K \otimes K \otimes K \otimes K \otimes \mathbb{C}^n$, and its range is naturally identified with $\mathbb{C}^n$. $\tilde{R}_{1n}$ is a unitary on H_n commuting with both E_{1n} and E_{2n} , whose restriction to $D_n = E_{1n}E_{2n}H_n$ is R_{1n} . In Theorem 1.1: Under $H_n = K \otimes K \otimes K \otimes K \otimes \mathbb{C}^n$, $\tilde{R}_{1n} = I_K \otimes I_K \otimes I_K \otimes I_K \otimes U_n$; under the natural identification $D_n \cong \mathbb{C}^n$, its restriction is $R_{1n} = U_n$. $\tilde{R}_{2n}$ is a unitary on H_n commuting with both E_{1n} and E_{2n} , whose restriction to $D_n = E_{1n}E_{2n}H_n$ is R_{2n} . In Theorem 1.1: Under $H_n = K \otimes K \otimes K \otimes K \otimes \mathbb{C}^n$, $\tilde{R}_{2n} = I_K \otimes I_K \otimes I_K \otimes I_K \otimes V_n$; under the natural identification $D_n \cong \mathbb{C}^n$, its restriction is $R_{2n} = V_n$. $\mathcal{A}_{1n} \subset \mathcal{B}(H_n)$ is a unital von Neumann algebra commuting with $\mathcal{A}_{2n}$. In Theorem 1.1: On the hypothetical infinite subsequence admitting the implementing unitaries u_n with $\|u_n - I\| \rightarrow 0$, $\mathcal{A}_{1n}$ is the von Neumann algebra generated by $\{\Sigma_n(g, e) : g \in F\}$. $\mathcal{A}_{2n} \subset \mathcal{B}(H_n)$ is a unital von Neumann algebra commuting with $\mathcal{A}_{1n}$. In Theorem 1.1: On the hypothetical infinite subsequence admitting the implementing unitaries u_n with $\|u_n - I\| \rightarrow 0$, $\mathcal{A}_{2n}$ is the von Neumann algebra generated by $\{\Sigma_n(e, g) : g \in F\}$. For $i \in \{1, 2\}$, $T_{in} \in \mathcal{A}_{in}$ and $\|T_{in} - E_{in}\tilde{R}_{in}\| \rightarrow 0$ as $n \rightarrow \infty$ along the sequence under consideration. In Theorem 1.1: On the hypothetical infinite subsequence admitting $u_n M_n u_n^* \subset N_n$ with $\|u_n - I\| \rightarrow 0$, choose a weak operator cluster point B_{in} of $\Sigma_n(g_{ij})$ as $j \rightarrow \infty$ and set $T_{in} = t^{-1}B_{in}$, where $g_{1j} = (s_j, e)$ and $g_{2j} = (e, s_j)$. $\hat{R}_{1n}$ is chosen for all sufficiently large n as a unitary on D_n , commuting with $\hat{R}_{2n}$, with $\|\hat{R}_{1n} - R_{1n}\| \rightarrow 0$. $\hat{R}_{2n}$ is chosen for all sufficiently large n as a unitary on D_n , commuting with $\hat{R}_{1n}$, with $\|\hat{R}_{2n} - R_{2n}\| \rightarrow 0$.

Proof. We first replace the intended projections by nearby projections in the commuting algebras. Set $\delta_{in} = \|T_{in} - E_{in}\tilde{R}_{in}\|$. Since $(E_{in}\tilde{R}_{in})^*(E_{in}\tilde{R}_{in}) = E_{in}$,

$$d_{in} := \|T_{in}^*T_{in} - E_{in}\| \leq \delta_{in}(2 + \delta_{in}) \rightarrow 0.$$

For sufficiently large n , take the spectral projection

$$Q_{in} = \mathbf{1}_{(1/2, \infty)}(T_{in}^*T_{in}) \in \mathcal{A}_{in}.$$

The spectrum of $T_{in}^*T_{in}$ lies in $[0, d_{in}] \cup [1 - d_{in}, 1 + d_{in}]$. This follows from the resolvent bound for a self-adjoint operator within d_{in} of a projection. Functional calculus therefore gives

$$\|Q_{in} - T_{in}^*T_{in}\| \leq d_{in}, \quad \|Q_{in} - E_{in}\| \leq 2d_{in} \rightarrow 0.$$

Since the two algebras commute, $P_n = Q_{1n}Q_{2n}$ is a projection, and

$$\|P_n - E_n\| \leq \|Q_{1n} - E_{1n}\| + \|Q_{2n} - E_{2n}\| \rightarrow 0.$$

We identify this perturbed corner with D_n by a small unitary. The operator

$$S_n = P_n E_n + (I - P_n)(I - E_n) = I + (P_n - E_n)(2E_n - I)$$

tends to the identity in norm and satisfies $S_n E_n = P_n S_n$. It is invertible for large n . Its absolute value commutes with E_n , so its unitary polar part $J_n = S_n(S_n^* S_n)^{-1/2}$ satisfies

$$J_n E_n J_n^* = P_n, \quad \|J_n - I\| \rightarrow 0. \quad (4.3)$$

In particular, $P_n H_n$ has exactly the dimension of D_n .

The compressions $P_n T_{1n} P_n$ and $P_n T_{2n} P_n$ commute, including across adjoints. This needs a check because Q_{in} need not commute with T_{in} . Suppress n and put $A = Q_1 T_1 Q_1 \in \mathcal{A}_1$ and $B = Q_2 T_2 Q_2 \in \mathcal{A}_2$. The two compressions are AQ_2 and $Q_1 B$. Since $AQ_1 = Q_1 A = A$, $BQ_2 = Q_2 B = B$, and every element of $\mathcal{A}_1$ commutes with every element of $\mathcal{A}_2$, both orders of their product equal AB . Replacing A or B by its adjoint proves the other commutations.

Transfer these compressions to D_n using (4.3):

$$C_{in} = E_n J_n^* T_{in} J_n E_n|_{D_n}.$$

The pair still commutes across adjoints. Moreover,

$$\|C_{in} - R_{in}\| \leq \delta_{in} + 2\|J_n - I\| \longrightarrow 0,$$

by comparing T_{in} with $E_{in} \tilde{R}_{in}$ before conjugation. Thus C_{in} is invertible for large n and $\|C_{in}^* C_{in} - I_{D_n}\| \rightarrow 0$. Its unitary polar part

$$\hat{R}_{in} = C_{in} (C_{in}^* C_{in})^{-1/2}$$

remains close to R_{in} . The unital C^* -algebras generated by C_{1n} and C_{2n} commute; continuous functional calculus puts each polar part in its respective algebra. Hence the two polar unitaries commute, as required. $\square$

4.3 Proof of the main theorem

Apply the construction of Section 3 to $D = \mathbb{C}^n$ and the clock-shift pair of Lemma 4.1. Use subscripts n for the resulting objects $L_n, H_n, a_n, b_n, f_n, W_n, M_n, N_n$. All these Hilbert spaces are separable. By Proposition 3.1,

$$\gamma(M_n, N_n) \leq C\nu_n \longrightarrow 0. \quad (4.4)$$

Remark (notation). For each integer $n \geq 3$, $L_n = K \otimes K \otimes \mathbb{C}^n$ is the separable Hilbert space obtained by substituting the clock-shift pair into the construction. For each integer $n \geq 3$, $a_n(g) = \sum_{k \in \mathbb{Z}} A_k(g) \otimes I_K \otimes U_n^k$ for $g \in F$, with the series converging absolutely in norm. For each integer $n \geq 3$, $b_n : F \rightarrow \mathcal{B}(L_n)$ is given by $b_n(g) = \sum_{k \in \mathbb{Z}} I_K \otimes A_k(g) \otimes V_n^k$, where $L_n = K \otimes K \otimes \mathbb{C}^n$. For each integer $n \geq 3$, $f_n(x) = a_n(x_1) b_n(x_2)$ for $x = (x_1, x_2) \in G$, using the construction with $D = \mathbb{C}^n$ and inputs U_n, V_n . For each integer $n \geq 3$, W_n is the diagonal unitary obtained from W by taking $D = \mathbb{C}^n$, $U = U_n$, and $V = V_n$.

Proof of Theorem 1.1. We show that no infinite subsequence admits unitaries u_n with

$$u_n M_n u_n^* \subset N_n, \quad \|u_n - I\| \longrightarrow 0. \quad (4.5)$$

Suppose otherwise and work on that subsequence. Define, for $g \in G$,

$$Z_n(g) = W_n(\lambda(g) \otimes I_{L_n}) W_n^*, \quad \Sigma_n(g) = u_n Z_n(g) u_n^* (\rho(g) \otimes I_{L_n}).$$

The first factor in the expression for $\Sigma_n(g)$ lies in N_n , by (4.5). Thus the two representations being multiplied have commuting ranges, and Σ_n is a unitary representation of G . Proposition 3.1 gives the uniform estimate

$$\sup_{g \in G} \|\Sigma_n(g) - \lambda(g) \rho(g) \otimes f_n(g)\| \leq \beta_n, \quad \beta_n = C\nu_n + 2\|u_n - I\| \longrightarrow 0. \quad (4.6)$$

Let $\mathcal{A}_{1n}$ and $\mathcal{A}_{2n}$ be the von Neumann algebras generated by the restrictions of Σ_n to $F \times \{e\}$ and $\{e\} \times F$, respectively. These algebras commute.

We will apply Lemma 4.2 by taking weak limits of the generators in each factor. Under $S = K \otimes K$, write

$$H_n = K \otimes K \otimes K \otimes K \otimes \mathbb{C}^n.$$

Let E_{1n} use p_e in the first and third slots, and let E_{2n} use p_e in the second and fourth slots, with identities in all other slots. These projections commute, and $E_n = E_{1n}E_{2n}$ has range naturally identified with $\mathbb{C}^n$. Let $\tilde{R}_{1n}$ and $\tilde{R}_{2n}$ act as U_n, V_n , respectively, on the final slot and as the identity elsewhere.

Set $g_{1j} = (s_j, e)$ and $g_{2j} = (e, s_j)$. On the i -th factor of S , the operator $\lambda(g_{ij})\rho(g_{ij})$ sends δ_x to $\delta_{s_j x s_j^{-1}}$. For each fixed nonidentity word x , this conjugate leaves every finite subset of F as $j \rightarrow \infty$: for large j , the letter s_j occurs neither in x nor in the finitely many words under consideration. Hence the conjugation operators converge weakly to p_e in that slot. Combining this observation with (3.2) gives, for each fixed n and i ,

$$\lambda(g_{ij})\rho(g_{ij}) \otimes f_n(g_{ij}) \xrightarrow{j \rightarrow \infty}^{\text{WOT}} tE_{in}\tilde{R}_{in}. \quad (4.7)$$

Here tensor products of bounded weakly convergent sequences on separate slots converge weakly, as follows first on elementary tensors and then by density. The finite spectral decompositions used in (3.2) justify the matrix-valued factor.

For each i, n , choose a weak operator cluster point B_{in} of $\Sigma_n(g_{ij})$ as $j \rightarrow \infty$. Such a point exists by weak operator compactness of the closed unit ball of $\mathcal{B}(H_n)$, and belongs to $\mathcal{A}_{in}$. Norm balls are weak operator closed, so (4.6) and (4.7) imply

$$\|B_{in} - tE_{in}\tilde{R}_{in}\| \leq \beta_n.$$

Thus $T_{in} = t^{-1}B_{in} \in \mathcal{A}_{in}$ satisfies (4.2). No product of weak limits is used: commutation comes from membership in the two commuting algebras. Lemma 4.2 now produces commuting unitary matrices on $\mathbb{C}^n$ approaching U_n, V_n in norm. This contradicts Lemma 4.1, proving that (4.5) cannot hold on any infinite subsequence.

It follows that some $\epsilon_0 > 0$ bounds $\|u - I_{H_n}\|$ from below for every implementing unitary and all sufficiently large n . Otherwise one could choose increasing n_j and implementing unitaries with distance less than $1/j$, contrary to what we have just proved. Together with (4.4), this is Theorem 1.1. $\square$

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