

# A Sharp Threshold Bound for Monotone Graph Properties

OpenAI

September 25, 2026

## Abstract

We prove the Friedgut–Kalai sharp-threshold conjecture. For every integer  $n \geq 2$ , every nontrivial increasing family of graphs on  $n$  vertices invariant under all vertex permutations, and every  $0 < \varepsilon < 1/2$ , the edge probabilities at which its probability equals  $\varepsilon$  and  $1 - \varepsilon$  differ by at most  $C \log(1/(2\varepsilon))/(\log n)^2$ , for a universal constant  $C$ .

## 1 Introduction

For an integer  $n \geq 2$ , write  $[n] = \{1, \dots, n\}$  and let  $E_n = \binom{[n]}{2}$  be the edge set of the complete simple graph on  $n$  vertices. We identify a graph with an element of  $\{0, 1\}^{E_n}$ . A *graph property* is a family  $\mathcal{P} \subseteq \{0, 1\}^{E_n}$  invariant under every permutation of  $[n]$ . It is *increasing*, or *monotone*, if adding edges preserves membership, and *nontrivial* if it is neither empty nor the whole space.

*Remark (notation).*  $\mathcal{P} \subseteq \{0, 1\}^{E_n}$  is a graph family invariant under every permutation of  $[n]$ . In Theorem 1.1:  $\mathcal{P}$  is increasing under addition of edges and satisfies  $\emptyset \neq \mathcal{P} \neq \{0, 1\}^{E_n}$ .

Write  $\mu_p(\mathcal{P})$  for the probability that a graph whose edges are present independently with probability  $p$  belongs to  $\mathcal{P}$ . For  $0 < a < 1$ , define

$$p_a(\mathcal{P}) = \inf\{p \in [0, 1] : \mu_p(\mathcal{P}) \geq a\}.$$

All logarithms in this paper are natural. In particular,  $n$  always counts vertices, not edge coordinates.

*Remark (notation).* For  $0 < a < 1$ ,  $p_a(\mathcal{P}) = \inf\{p \in [0, 1] : \mu_p(\mathcal{P}) \geq a\}$  whenever this defining set is nonempty. In Theorem 1.1:  $p_a(\mathcal{P}) \in (0, 1)$  is the unique solution of  $\mu_{p_a(\mathcal{P})}(\mathcal{P}) = a$  for every  $0 < a < 1$ .

**Theorem 1.1** (Friedgut–Kalai sharp-threshold conjecture). *For every integer  $n \geq 2$ , every nontrivial increasing family  $\mathcal{P}$  of graphs on  $n$  vertices invariant under all vertex permutations, and every  $0 < \varepsilon < 1/2$ ,*

$$p_{1-\varepsilon}(\mathcal{P}) - p_\varepsilon(\mathcal{P}) \leq \frac{2^{19}}{(\log n)^2} \log \frac{1}{2\varepsilon}. \quad (1.1)$$

The influence theorem of Kahn, Kalai, and Linial [6] connected Fourier analysis with the sensitivity of Boolean functions to individual coordinates. Bourgain, Kahn, Kalai, Katznelson, and Linial extended the influence theorem to product spaces [2]. Friedgut and Kalai combined that extension with the Margulis–Russo probability-derivative identity to show that every increasing family invariant under a transitive coordinate group has a narrow transition [5, Section 2]. For graph properties, vertex permutations act transitively on the edge coordinates, and their theorem gives the universal width bound  $C \log(1/(2\varepsilon))/\log n$  [5, Theorem 1.1]. They conjectured the

denominator  $(\log n)^2$  with the same dependence on  $\varepsilon$  [5, Conjecture 1.2]. Theorem 1.1 resolves that conjecture positively; its numerical constant is only a convenient explicit choice.

The squared logarithm is the optimal order in  $n$  when  $\varepsilon$  is fixed. Friedgut and Kalai already pointed to the property of containing a clique with order proportional to  $\log n$ : its transition width can be of order  $(\log n)^{-2}$  [5, p. 2994]. Bourgain and Kalai went beyond coordinate transitivity by using the action of the symmetry group on sets of coordinates [3]. For every fixed  $\eta > 0$  and  $0 < \varepsilon < 1/2$ , their graph-property bound is

$$p_{1-\varepsilon}(\mathcal{P}) - p_\varepsilon(\mathcal{P}) \leq C_{\eta,\varepsilon}(\log n)^{-2+\eta}.$$

Thus their result approached the conjectured exponent arbitrarily closely. Further progress through Fourier estimates is described below; Friedgut gives a broader account of the influence method in [4].

Two companion papers address complementary threshold-location questions: comparison of integral and fractional covers of increasing families [8, Theorem 1.1], and comparison of graph-containment thresholds with expected subgraph counts [9, Theorem 1.1]. The present theorem controls the width of a transition for arbitrary increasing graph properties. Its proof does not use either comparison.

Here is the stronger estimate that drives the proof. For a Boolean function  $f : \{0, 1\}^{E_n} \rightarrow \{0, 1\}$ , define its *pivotal influence* at edge  $e$  and its *total influence* as follows. Here  $x_{e \leftarrow b}$  denotes  $x$  with coordinate  $e$  set to  $b \in \{0, 1\}$ , and the probability is over all coordinates other than  $e$ .

$$I_{e,p}(f) = \Pr_p(f(x_{e \leftarrow 1}) \neq f(x_{e \leftarrow 0})), \quad I_p(f) = \sum_{e \in E_n} I_{e,p}(f).$$

*Remark (notation).*  $f$  is a Boolean function  $f : \{0, 1\}^{E_n} \rightarrow \{0, 1\}$ . In Theorem 1.1:  $f = \mathbf{1}_{\mathcal{P}}$ , where  $\mathcal{P}$  is the nontrivial increasing graph property under consideration. In Proposition 3.2, Lemma 3.3, Theorem 1.2:  $f$  is invariant under every permutation of  $[n]$ ; monotonicity is not assumed. In Lemma 3.1:  $f$  is an arbitrary Boolean function on  $\{0, 1\}^{E_n}$ ; neither symmetry nor monotonicity is assumed.  $x$  denotes an argument configuration on the coordinate set of the function being evaluated; in  $x_{e \leftarrow b}$  it is a graph configuration in  $\{0, 1\}^{E_n}$ . In Lemma 2.1: In the Fourier characters,  $x \in \{0, 1\}^J$ ; as an argument of  $\Delta_i g$ ,  $x$  ranges over the remaining coordinates  $J \setminus \{i\}$ . In Lemma 3.1: In  $f_y(x) = f(x, y)$ ,  $x \in \{0, 1\}^F$  is the free-coordinate configuration, while  $y \in \{0, 1\}^H$  specifies the exterior coordinates. In the notation  $x_{e \leftarrow b}$ ,  $b \in \{0, 1\}$  is the value assigned to coordinate  $e$ . In Lemma 2.1: In the scalar two-point inequality,  $b$  is an arbitrary real scalar. In tensorization, the same letter locally denotes the real-valued coefficient function on  $\{0, 1\}^{J \setminus \{i\}}$  in  $g = a + b\chi_i$ ; this is a function of the remaining coordinates.

**Theorem 1.2** (Variance and influence). *For every integer  $n \geq 2$ , every Boolean function  $f$  on  $E_n$  invariant under vertex permutations, and every  $0 < p < 1$ ,*

$$\mathrm{Var}_p(f) \leq \frac{2^{17}}{(\log n)^2} I_p(f). \quad (1.2)$$

*No monotonicity assumption is needed.*

At the unbiased measure, Kelman, Kindler, Lifshitz, Minzer, and Safra proved the graph-symmetric estimate

$$I_{1/2}(f) \geq c \frac{(\log n)^2}{(\log \log n)^2} \mathrm{Var}_{1/2}(f)$$

for all sufficiently large  $n$  and an absolute  $c > 0$  [7, arXiv version 2, Section 7.4]. This is an influence estimate at  $p = 1/2$ ; a threshold-width bound requires control throughout the transition interval. Theorem 1.2 removes the iterated-logarithm loss and holds uniformly for every  $0 < p < 1$ , including for nonmonotone properties.

Random restrictions and symmetry of Fourier coefficients are central to the arguments of Bourgain–Kalai and Kelman–Kindler–Lifshitz–Minzer–Safra [3, 7]. Here the restriction is chosen through the vertices: leave free all edges touching a random block  $B$  of about  $\sqrt{n}$  vertices, while conditioning on every other edge. Permutations inside  $B$  survive every such conditioning. Their edge orbits bound each conditional influence, to which an elementary Fourier estimate applies. The resulting estimate is linear in total influence, so it can be averaged over the conditioned coordinates. Section 2 proves this estimate directly, including its dependence on the bias.

*Remark (notation).* The notation  $B$  is local to its context, denoting either the auxiliary function  $T_\rho b$  or an  $m$ -element vertex block. In Lemma 2.1: In the tensorization proof,  $B = T_\rho b$  is a real function on the coordinates other than  $i$ , where  $g = a + b\chi_i$ . In Proposition 3.2:  $B \in \binom{[n]}{m}$ , with integer  $3 \leq m \leq n$ . In Lemma 3.3: The proof uses  $B \in \binom{[n]}{m}$  with  $m = \lfloor \sqrt{n} \rfloor$ .

The second ingredient regards each edge set indexing a Fourier coefficient as a graph. Any nonempty such graph with at most  $k^2/2$  edges has a nonisolated vertex of degree at most  $k$ . With sufficient probability,  $B$  meets its nonisolated vertices only at that vertex. The restriction then leaves between one and  $k$  edges of this graph free. This turns control of conditional Fourier degrees up to  $k$  into control of original degrees up to  $k^2/2$ . Taking  $k$  proportional to  $\sqrt{p(1-p)} \log n$  yields (1.2); the bias factor cancels in the estimate for the remaining Fourier degrees. Section 4 then integrates the probability derivative to obtain the full quantile bound.

*Remark (notation).*  $k$  denotes a positive real degree cutoff, with comparisons to integer cardinalities interpreted literally rather than requiring  $k$  to be an integer. In Lemma 2.1:  $k = \sigma \log m/64$ , where  $m \geq 2$  is the real influence comparison parameter. In Lemma 3.1:  $k$  is any positive real number, with no required relation to an influence comparison parameter. In Proposition 3.2:  $k = \sigma \log m/64$ , where  $m$  is the integer block size satisfying  $3 \leq m \leq n$ . In Lemma 3.3, Theorem 1.2: When  $t = \log n \geq 16$ ,  $m = \lfloor \sqrt{n} \rfloor$  and  $k = \sigma \log m/64$ ; the original Fourier levels are split at  $k^2/2$ .

## 2 A low-degree estimate on biased products

We bound the mass of low-degree Fourier coefficients for Boolean functions whose individual influences are small compared with their total influence. The bound is linear in total influence, so it can be averaged over the conditioned coordinates in Section 3.

Let  $J$  be a finite coordinate set, let  $0 < p < 1$ , and put  $\sigma = \sqrt{p(1-p)}$ . Expectations and norms in this section are with respect to the product Bernoulli- $p$  measure on the indicated coordinates; in particular,  $\|g\|_q = (\mathbb{E}|g|^q)^{1/q}$ . Define

$$\chi_i(x) = \frac{x_i - p}{\sigma}, \quad \chi_S(x) = \prod_{i \in S} \chi_i(x), \quad \chi_\emptyset = 1.$$

Each  $\chi_i$  has mean zero and second moment one. Independence therefore makes the  $2^{|J|}$  functions  $\chi_S$ ,  $S \subseteq J$ , orthonormal. They form a basis of the  $2^{|J|}$ -dimensional space of real functions on  $\{0, 1\}^J$ . Thus, with  $\hat{g}(S) = \mathbb{E}[g\chi_S]$ ,

$$g = \sum_{S \subseteq J} \hat{g}(S) \chi_S, \quad \mathbb{E}g^2 = \sum_{S \subseteq J} \hat{g}(S)^2, \quad \text{Var}(g) = \sum_{\emptyset \neq S \subseteq J} \hat{g}(S)^2. \quad (2.1)$$

*Remark (notation).*  $S$  is a subset of the coordinate set indexing a product Fourier character or coefficient. In Lemma 2.1:  $S \subseteq J$ ; the low-degree sum ranges over  $1 \leq |S| \leq k$ , interpreted literally for integer  $|S|$ . In Lemma 3.1, Proposition 3.2, Lemma 3.3, Theorem 1.2: The original graph Fourier indices satisfy  $S \subseteq E_n$ , so  $S$  can also be viewed as the edge set of a simple graph on  $[n]$ . In Lemma 3.3: The support-selection argument fixes each nonempty  $S$  with  $|S| \leq k^2/2$  separately; no single block is required to capture all such supports.

For  $i \in J$ , write  $\Delta_i g(x) = g(x_{J \setminus \{i\}}, 1) - g(x_{J \setminus \{i\}}, 0)$ , a function on the remaining coordinates. Integrating just coordinate  $i$  gives

$$\mathbb{E}_i[g\chi_i] = \frac{p(1-p)}{\sigma} (g(x_{J \setminus \{i\}}, 1) - g(x_{J \setminus \{i\}}, 0)) = \sigma \Delta_i g.$$

Consequently, for  $A \subseteq J \setminus \{i\}$ ,  $\widehat{g}(A \cup \{i\}) = \sigma \widehat{\Delta_i g}(A)$ , where the latter Fourier coefficient is taken on  $J \setminus \{i\}$ . For Boolean  $h : \{0, 1\}^J \rightarrow \{0, 1\}$ , set

$$I_i(h) = \Pr(\Delta_i h \neq 0), \quad I(h) = \sum_{i \in J} I_i(h).$$

Since  $\Delta_i h \in \{-1, 0, 1\}$ , Parseval on the remaining coordinates and then summation over  $i$  give

$$\sum_{S \ni i} \widehat{h}(S)^2 = \sigma^2 I_i(h), \quad \sum_{S \subseteq J} |S| \widehat{h}(S)^2 = \sigma^2 I(h). \quad (2.2)$$

These identities do not require monotonicity.

*Remark (notation).*  $A$  is locally bound notation denoting either a Fourier index set on the indicated coordinates or the transformed function  $T_\rho a$ , according to its surrounding definition. In Lemma 2.1: In the derivative coefficient identities and sums,  $A \subseteq J \setminus \{i\}$ . Within the tensorization argument,  $A = T_\rho a$  is instead a real function on the coordinates  $J \setminus \{i\}$ . In Lemma 3.1:  $A \subseteq F$  indexes the Fourier coefficients of  $f_y$  on the free coordinates; the displayed mass sum restricts it to  $1 \leq |A| \leq k$ , while the coefficient expansion holds for every  $A \subseteq F$ .

The next estimate follows the hypercontractive influence method of Kahn, Kalai, and Linial [6]. We prove the biased two-point inequality directly and use the standard product argument of Bonami [1, Chapter III, Lemma 1], keeping the dependence on the edge probability explicit.

**Lemma 2.1.** *Let  $m \geq 2$  be real and let  $h : \{0, 1\}^J \rightarrow \{0, 1\}$  satisfy  $I_i(h) \leq I(h)/m$  for every  $i \in J$ . If*

$$k = \frac{\sigma \log m}{64},$$

*then*

$$\sum_{1 \leq |S| \leq k} \widehat{h}(S)^2 \leq m^{-1/4} I(h). \quad (2.3)$$

*Every degree cutoff is interpreted literally for integer degrees  $|S|$ ; in particular, the sum is empty when  $k < 1$ .*

*Remark (notation).*  $m \geq 2$  is a numerical parameter whose admissible values and choice depend on its local use. In Lemma 2.1:  $m$  is any real number at least 2 satisfying the stipulated comparison  $I_i(h) \leq I(h)/m$  for every  $i \in J$ . In Proposition 3.2:  $m$  is an integer with  $3 \leq m \leq n$  and is the cardinality of the vertex block. In Lemma 3.3:  $m = \lfloor \sqrt{n} \rfloor$ , under the assumption  $\log n \geq 16$ . In Theorem 1.2: In the proof branch  $\log n \geq 16$ ,  $m = \lfloor \sqrt{n} \rfloor$ ; the branch  $2 \leq n < e^{16}$  does not use  $m$ .

*Proof.* We first prove a product norm bound. Put  $\rho = \sigma/4$ , so  $0 < \rho \leq 1/8$ , and define the linear operator  $T_\rho$  by  $T_\rho \chi_S = \rho^{|S|} \chi_S$ . For one coordinate, the bound  $|\chi_i| \leq \sigma^{-1}$  and the second moment identity imply

$$|\mathbb{E} \chi_i^3| \leq \sigma^{-1}, \quad \mathbb{E} \chi_i^4 \leq \sigma^{-2}.$$

For real  $a, b$ , expansion and  $2|ab^3| \leq a^2b^2 + b^4$  yield

$$\begin{aligned} \mathbb{E}(a + \rho b \chi_i)^4 &\leq a^4 + 6\rho^2 a^2 b^2 + \frac{4\rho^3}{\sigma} |ab^3| + \frac{\rho^4}{\sigma^2} b^4 \\ &\leq a^4 + \frac{13\sigma^2}{32} a^2 b^2 + \frac{9\sigma^2}{256} b^4 \leq (a^2 + b^2)^2, \end{aligned}$$

where the last inequality uses  $\sigma^2 \leq 1/4$ .

To tensorize this inequality, induct on  $|J|$ , the empty product being immediate. Write  $g = a + b\chi_i$  with  $a, b$  functions of the other coordinates, and set  $A = T_\rho a$ ,  $B = T_\rho b$ , using the operator on those coordinates. Apply the scalar inequality pointwise and then the triangle inequality in  $L^2$  to obtain

$$\|T_\rho g\|_4^2 \leq \|A^2 + B^2\|_2 \leq \|A\|_4^2 + \|B\|_4^2 \leq \|a\|_2^2 + \|b\|_2^2 = \|g\|_2^2.$$

The penultimate inequality is the induction hypothesis. We have proved

$$\|T_\rho g\|_4 \leq \|g\|_2 \tag{2.4}$$

on every finite product.

*Remark (notation).* In  $p_a(\mathcal{P})$ ,  $a \in (0, 1)$  is the probability level in the defining inequality  $\mu_p(\mathcal{P}) \geq a$ . In Lemma 2.1: In the scalar two-point inequality,  $a$  is an arbitrary real scalar. In tensorization, the same letter locally denotes the real-valued coefficient function on  $\{0, 1\}^{J \setminus \{i\}}$  in  $g = a + b\chi_i$ ; this is a function of the remaining coordinates.

For any real  $d \geq 0$ , let  $P_{\leq d}$  be the orthogonal projection onto the span of the characters of degree at most  $d$ . If  $u = P_{\leq d} u$ , the operator  $T_\rho$  is invertible and Parseval gives

$$\|T_\rho^{-1} u\|_2^2 = \sum_{|S| \leq d} \rho^{-2|S|} \widehat{u}(S)^2 \leq \rho^{-2d} \|u\|_2^2.$$

Applying (2.4) to  $T_\rho^{-1} u$  proves  $\|u\|_4 \leq \rho^{-d} \|u\|_2$ . Duality in this finite-dimensional subspace, followed by Hölder's inequality, now gives for every real function  $v$

$$\|P_{\leq d} v\|_2 = \sup_{\substack{u = P_{\leq d} u \\ \|u\|_2 = 1}} |\mathbb{E}[vu]| \leq \rho^{-d} \|v\|_{4/3}. \tag{2.5}$$

*Remark (notation).* The notation  $u$  is local to its context, denoting a real function in the projection argument or a vertex in the graph arguments. In Lemma 2.1:  $u$  is a real function on  $\{0, 1\}^J$  satisfying  $u = P_{\leq d} u$ ; in the supremum defining the projection norm it also satisfies  $\|u\|_2 = 1$ . In Proposition 3.2: In  $\{\{u, w\} : u \in B\}$ ,  $u$  ranges over all vertices of  $B$ . In Lemma 3.3: For each fixed nonempty  $S \subseteq E_n$  with  $|S| \leq k^2/2$ , choose  $u \in V(S)$  with  $d_S(u) \leq 2|S|/|V(S)|$ . This choice may depend on  $S$  and is made before choosing the block;  $B \cap V(S) = \{u\}$  is an event, rather than a condition defining  $u$ .  $v$  is local notation for either a real function or a number of nonisolated vertices, according to its scope. In Lemma 2.1:  $v$  is an arbitrary real function on  $\{0, 1\}^J$  to which  $P_{\leq d}$  is applied. In Lemma 3.3: For the fixed nonempty edge set  $S \subseteq E_n$ ,  $v = |V(S)|$  is the number of its nonisolated vertices.

If  $k < 1$ , (2.3) is immediate, so assume  $k \geq 1$ . Since  $|\Delta_i h|$  is the indicator of a pivotal event,  $\|\Delta_i h\|_{4/3} = I_i(h)^{3/4}$ . Every nonempty Fourier set is counted at least once when summing over its coordinates. The derivative coefficient identity and (2.5), applied on  $J \setminus \{i\}$  with  $d = k$ , therefore show

$$\begin{aligned} \sum_{1 \leq |S| \leq k} \widehat{h}(S)^2 &\leq \sum_{i \in J} \sum_{\substack{S \ni i \\ |S| \leq k}} \widehat{h}(S)^2 \\ &= \sigma^2 \sum_{i \in J} \sum_{\substack{A \subseteq J \setminus \{i\} \\ |A| \leq k-1}} \widehat{\Delta_i h}(A)^2 \\ &\leq \sigma^2 \rho^{-2k} \sum_{i \in J} I_i(h)^{3/2}. \end{aligned}$$

The function  $s \mapsto s \log(4/s)$  is increasing on  $(0, 1/2]$ , since its derivative is  $\log(4/s) - 1 > 0$ . Hence  $\sigma \log(4/\sigma) \leq \frac{1}{2} \log 8 < 2$ , and

$$\rho^{-2k} = m^{\sigma \log(4/\sigma)/32} \leq m^{1/16}.$$

If  $I(h) \leq m^{1/4}$ , the hypothesis implies

$$\sum_i I_i(h)^{3/2} \leq \sqrt{\frac{I(h)}{m}} I(h) \leq m^{-3/8} I(h).$$

The preceding estimates bound the desired sum by  $\sigma^2 m^{-5/16} I(h) \leq m^{-1/4} I(h)$ , including when  $I(h) = 0$ . If  $I(h) > m^{1/4}$ , Parseval and Booleanity instead give

$$\sum_{1 \leq |S| \leq k} \widehat{h}(S)^2 \leq \mathbb{E} h^2 \leq 1 < m^{-1/4} I(h).$$

This proves the estimate for every  $0 < p < 1$ , without a lower bound on  $\sigma$ .  $\square$

*Remark (notation).*  $s$  is a positive scalar whose meaning is local to the argument. In Lemma 2.1:  $s$  is a real argument in  $(0, 1/2]$  for the function  $s \mapsto s \log(4/s)$ . In Lemma 3.3:  $s = |S|$  is the positive integer number of edges of the fixed support graph, with  $1 \leq s \leq k^2/2$ .

### 3 Restricting to the edges incident with a vertex block

Let  $E_n = \binom{[n]}{2}$  be the edge set of the complete graph on  $[n]$ , and let  $f: \{0, 1\}^{E_n} \rightarrow \{0, 1\}$  be invariant under vertex permutations. Fix  $p \in (0, 1)$  and put  $\sigma = \sqrt{p(1-p)}$ . All Fourier coefficients, variances, and influences in this section use this bias. Monotonicity is not needed.

The purpose of a vertex block is to connect two different notions of degree. A restriction to the edges incident with the block retains enough symmetry to apply Lemma 2.1. If the block meets the nonisolated vertices of a Fourier support at just one vertex  $u$ , then the surviving Fourier degree is the graph degree of  $u$ . An  $s$ -edge graph with  $s \geq 1$  has a nonisolated vertex of degree at most  $\sqrt{2s}$ . Thus a bound for conditional Fourier degrees up to  $k$  will control original degrees up to  $k^2/2$ .

The restriction identity is the biased-product form of the standard Fourier restriction identity; compare [7, arXiv version 2, Lemma 2.7]. We include its short proof. The next identities hold for every Boolean  $f$ , without a symmetry assumption.

**Lemma 3.1** (Exact averaging over a restriction). *Let  $F \subseteq E_n$ , put  $H = E_n \setminus F$ , and, for  $y \in \{0, 1\}^H$ , let  $f_y(x) = f(x, y)$  for  $x \in \{0, 1\}^F$ . If  $y$  has the Bernoulli- $p$  product law, then, for every real  $k > 0$ ,*

$$\mathbb{E}_y \sum_{\substack{A \subseteq F \\ 1 \leq |A| \leq k}} \widehat{f_y}(A)^2 = \sum_{\substack{S \subseteq E_n \\ 1 \leq |S \cap F| \leq k}} \widehat{f}(S)^2, \quad (3.1)$$

$$\mathbb{E}_y I(f_y) = \sum_{e \in F} I_e(f). \quad (3.2)$$

*Remark (notation).*  $y \in \{0, 1\}^H$  denotes the exterior assignment, held fixed when defining or evaluating the restricted function  $f_y$  and integrated out when an exterior expectation  $\mathbb{E}_y$  is taken. In Proposition 3.2: The exterior coordinate set is  $H_B = E_n \setminus F_B$ , consisting of the edges with both endpoints outside  $B$ . For each fixed exterior assignment  $y \in \{0, 1\}^H$ ,  $f_y : \{0, 1\}^F \rightarrow \{0, 1\}$  is the function  $x \mapsto f(x, y)$ , where  $(x, y)$  is the configuration on  $E_n = F \cup H$ . In Proposition 3.2: Here  $F = F_B = \{e \in E_n : e \cap B \neq \emptyset\}$  and  $H = H_B = E_n \setminus F_B$ ; for every exterior assignment  $y$ ,  $f_y$  is invariant under permutations supported on  $B$ . For  $A \subseteq F$  and  $y \in \{0, 1\}^H$ , where  $H = E_n \setminus F$ ,  $\widehat{f_y}(A) = \mathbb{E}_{x \sim \text{Bernoulli}(p)^{\otimes F}} [f(x, y) \chi_A(x)]$ ; this integrates only the free coordinates and retains its dependence on  $y$ .  $I(f_y) = \sum_{e \in F} \Pr_p(f_y(x_{e \leftarrow 1}) \neq f_y(x_{e \leftarrow 0}))$ , with  $y$  held fixed and each pivotal probability integrating only the coordinates in  $F \setminus \{e\}$ . In Proposition 3.2: The free coordinate set is  $F_B = \{e \in E_n : e \cap B \neq \emptyset\}$ , and  $I(f_y)$  is evaluated with both  $B$  and its exterior assignment  $y$  fixed before averaging.

*Proof.* Expand  $f$  in the product Fourier basis and integrate over  $x$ . For each  $A \subseteq F$  this gives the exact finite identity

$$\widehat{f_y}(A) = \sum_{D \subseteq H} \widehat{f}(A \cup D) \chi_D(y).$$

Orthonormality on the exterior coordinates therefore gives  $\mathbb{E}_y \widehat{f_y}(A)^2 = \sum_{D \subseteq H} \widehat{f}(A \cup D)^2$ . Summing over  $A$  proves (3.1). In particular, there is no constraint on the exterior Fourier degree  $|D|$ .

For  $e \in F$ , the pivotal event for  $f_y$  at  $e$  is the pivotal event for  $f$  at  $e$  with the exterior coordinates fixed to  $y$ . Averaging over  $y$  and the remaining free coordinates gives  $\mathbb{E}_y I_e(f_y) = I_e(f)$ ; summation proves (3.2).  $\square$

*Remark (notation).* For  $e \in F$  and each fixed exterior assignment  $y \in \{0, 1\}^H$ ,  $I_e(f_y)$  is the probability, over the remaining free coordinates with Bernoulli- $p$  product law, that changing coordinate  $e$  changes  $f_y$ .

**Proposition 3.2** (The block estimate). *Let  $3 \leq m \leq n$  be an integer, let  $B$  be uniform among the  $m$ -element subsets of  $[n]$ , and set*

$$F_B = \{e \in E_n : e \cap B \neq \emptyset\}, \quad k = \frac{\sigma \log m}{64}.$$

*Then*

$$\sum_{S \subseteq E_n} \Pr_B \{1 \leq |S \cap F_B| \leq k\} \widehat{f}(S)^2 \leq \frac{2m}{n} m^{-1/4} I(f). \quad (3.3)$$

*Proof.* Fix  $B$ , and fix an arbitrary assignment  $y$  to  $H_B = E_n \setminus F_B$ . The free set  $F_B$  contains all edges with at least one endpoint in  $B$ : both edges within  $B$  and edges from  $B$  to its complement. Every permutation supported on  $B$  fixes each edge of  $H_B$  individually, and hence fixes  $y$ . The

restricted function  $f_y$  is consequently invariant under this permutation group even if the graph encoded by  $y$  has no symmetry.

The free-edge orbits are precisely the  $\binom{m}{2}$  edges within  $B$  and, for each *fixed*  $w \in [n] \setminus B$ , the  $m$  edges  $\{\{u, w\} : u \in B\}$ . Every orbit has size at least  $m$ , since  $m \geq 3$ . The product law and the pivotal events are preserved by these permutations, so influences agree within each orbit. It follows that

$$I_e(f_y) \leq \frac{I(f_y)}{m} \quad (e \in F_B).$$

Lemma 2.1 applies separately to every  $y$ . Its right-hand side is linear in  $I(f_y)$ , so averaging and using Lemma 3.1 gives

$$\sum_{\substack{S \subseteq E_n \\ 1 \leq |S \cap F_B| \leq k}} \hat{f}(S)^2 \leq m^{-1/4} \sum_{e \in F_B} I_e(f).$$

Finally average over  $B$ . Each endpoint of a fixed edge belongs to  $B$  with probability  $m/n$ , whence  $\Pr_B(e \in F_B) \leq 2m/n$ . This proves (3.3).  $\square$

**Lemma 3.3** (Capturing low Fourier levels). *Suppose  $t = \log n \geq 16$ , and put*

$$m = \lfloor \sqrt{n} \rfloor, \quad k = \frac{\sigma \log m}{64}.$$

*Then*

$$\sum_{\substack{S \subseteq E_n \\ 1 \leq |S| \leq k^2/2}} \hat{f}(S)^2 \leq 4m^{-1/4} I(f). \quad (3.4)$$

*Proof.* Fix a nonempty  $S \subseteq E_n$  with  $s = |S| \leq k^2/2$ , viewed as the edge set of a simple graph. Let  $V(S)$  be its set of nonisolated vertices and write  $v = |V(S)|$ . Since  $s \leq \binom{v}{2} < v^2/2$ , its average degree over these vertices satisfies

$$\frac{2s}{v} < \sqrt{2s}.$$

Choose a vertex  $u \in V(S)$  whose degree  $d_S(u)$  is at most this average. Then

$$1 \leq d_S(u) \leq \sqrt{2s} \leq k, \quad v \leq 2s \leq k^2. \quad (3.5)$$

*Remark (notation).*  $d_S(u)$  is the degree of  $u$  in the simple graph with edge set  $S$ . In Lemma 3.3: For each fixed nonempty support  $S$  under consideration,  $u \in V(S)$  is chosen with  $d_S(u) \leq 2|S|/|V(S)|$ , so  $1 \leq d_S(u) \leq \sqrt{2|S|} \leq k$ .

Choose  $B$  uniformly among the  $m$ -element vertex sets. Conditioned on  $u \in B$ , each other specified vertex belongs to  $B$  with probability  $(m-1)/(n-1)$ . A union bound gives

$$\Pr_B\{B \cap V(S) = \{u\}\} \geq \frac{m}{n} \left(1 - (v-1) \frac{m-1}{n-1}\right). \quad (3.6)$$

On this event,  $S \cap F_B$  consists of *every* support edge incident with  $u$ , and contains no other support edge. Its size is therefore  $d_S(u)$ , between 1 and  $k$ ; it need not be 1. Figure 1 illustrates this distinction.

To bound the probability uniformly in  $p$ , note that  $k \leq t$  and

$$\frac{m-1}{n-1} \leq \frac{1}{\sqrt{n}}, \quad (v-1) \frac{m-1}{n-1} \leq t^2 e^{-t/2} \leq 256e^{-8} < \frac{1}{2}.$$

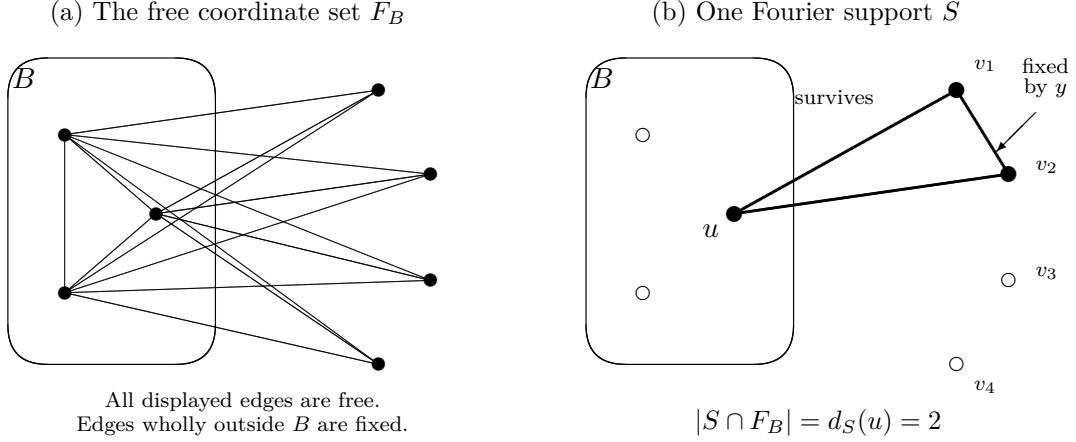


Figure 1: The block restriction and the one-vertex event. In (a),  $F_B$  contains every edge incident with  $B$ , including its internal edges; only edges with both endpoints outside  $B$  are fixed. In (b), only support edges are drawn, and filled vertices are the nonisolated vertices  $V(S)$ . The event  $B \cap V(S) = \{u\}$  leaves both support edges incident with  $u$  free. The exterior support edge is fixed, and its contribution is retained by the exact averaging identity (3.1).

Here  $t^2 e^{-t/2}$  is decreasing for  $t \geq 16$ . Consequently, for every support under consideration,

$$\Pr_B\{1 \leq |S \cap F_B| \leq k\} \geq \frac{m}{2n}. \quad (3.7)$$

The choice of  $u$  may depend on  $S$ . The lower bound (3.7) is applied to each nonnegative summand  $\hat{f}(S)^2$  separately; no block must capture all supports simultaneously. Comparing with (3.3) and cancelling  $m/(2n)$  proves (3.4). If  $k^2/2 < 1$ , the sum in (3.4) is empty and the conclusion holds without choosing a support.  $\square$

*Proof of Theorem 1.2.* First suppose  $t = \log n \geq 16$ , and choose  $m$  and  $k$  as in Lemma 3.3; in particular  $m \geq 3$  and  $k > 0$ . For the remaining Fourier degrees, the Dirichlet identity (2.2) gives

$$\sum_{|S| > k^2/2} \hat{f}(S)^2 \leq \frac{2}{k^2} \sum_S |S| \hat{f}(S)^2 = \frac{2\sigma^2}{k^2} I(f). \quad (3.8)$$

This remains valid for  $k^2/2 < 1$ , when the left side includes all nonconstant coefficients. Combining (2.1), (3.4), and (3.8) yields

$$\text{Var}(f) \leq \left(4m^{-1/4} + \frac{2\sigma^2}{k^2}\right) I(f). \quad (3.9)$$

For completeness, all constants can be bounded independently of  $p$  and  $n$ . Since  $m \geq \sqrt{n}/2$  and  $t \geq 16$ ,  $\log m \geq t/2 - \log 2 \geq t/3$ . Thus

$$\frac{2\sigma^2}{k^2} = \frac{8192}{(\log m)^2} \leq \frac{73728}{t^2}.$$

The cancellation of  $\sigma^2$  is exact, even when  $p$  is arbitrarily close to 0 or 1. Also,  $t^2 e^{-t/8}$  is nonincreasing for  $t \geq 16$ , so

$$4m^{-1/4} t^2 \leq 4 \cdot 2^{1/4} t^2 e^{-t/8} \leq 4 \cdot 2^{1/4} 256 e^{-2} < 256.$$

Thus the right-hand side of (3.9) is at most  $(73728 + 256)I(f)/t^2$ , which is at most  $2^{17}I(f)/t^2$ .

If instead  $2 \leq n < e^{16}$ , the same Fourier identities immediately give

$$\text{Var}(f) \leq \sigma^2 I(f) \leq \frac{1}{4} I(f) \leq \frac{64}{(\log n)^2} I(f).$$

Hence the single choice  $C_0 = 2^{17}$  works for every integer  $n \geq 2$  and every  $p \in (0, 1)$ , as asserted.  $\square$

## 4 From influence to threshold width

We pass from influence to transition width using the Russo–Margulis probability-derivative formula; see Russo [10, Lemma 3] and Friedgut–Kalai [5, Lemma 2.2]. We include the finite calculation with our pivotal-influence convention and then integrate the resulting differential inequality.

*Proof of Theorem 1.1.* Put  $f = \mathbf{1}_P$ ,  $\mu(p) = \mathbb{E}_p f$ , and  $C_0 = 2^{17}$ . We first verify the probability-derivative identity with the influence convention used here. If the edge probabilities are allowed to vary separately,  $\mathbb{E}f$  is a multilinear polynomial in those probabilities. Its partial derivative in the probability at  $e$  is the expectation of

$$\Delta_e f = f(x_{e \leftarrow 1}) - f(x_{e \leftarrow 0}).$$

For increasing  $f$ , this difference takes values in  $\{0, 1\}$ , so its expectation is  $I_{e,p}(f)$  when all other biases are  $p$ . Differentiating along the diagonal therefore gives

$$\mu'(p) = I_p(f). \tag{4.1}$$

This is the usual probability-derivative formula underlying the sharp-threshold method; the calculation above proves it directly in the present finite setting.

Since  $f$  is Boolean,  $\text{Var}_p(f) = \mu(p)(1 - \mu(p))$ . Theorem 1.2 and (4.1) imply

$$\mu'(p) \geq \frac{(\log n)^2}{C_0} \mu(p)(1 - \mu(p)) \quad (0 < p < 1). \tag{4.2}$$

Nontriviality and monotonicity give  $f(\emptyset) = 0$  and  $f(E_n) = 1$ . Both configurations have positive probability for every interior bias, hence  $0 < \mu(p) < 1$  there, while  $\mu(0) = 0$  and  $\mu(1) = 1$ . Equation (4.2) now gives  $\mu'(p) > 0$  throughout  $(0, 1)$ . Thus, by continuity, every quantile  $p_a$  is the unique interior point satisfying  $\mu(p_a) = a$ .

On the interval  $[p_\varepsilon, p_{1-\varepsilon}]$ , division in (4.2) is legitimate and gives

$$\frac{d}{dp} \log \frac{\mu(p)}{1 - \mu(p)} = \frac{\mu'(p)}{\mu(p)(1 - \mu(p))} \geq \frac{(\log n)^2}{C_0}.$$

Integration yields

$$p_{1-\varepsilon} - p_\varepsilon \leq \frac{2C_0}{(\log n)^2} \log \frac{1 - \varepsilon}{\varepsilon}.$$

Finally,  $4\varepsilon(1 - \varepsilon) \leq 1$  implies

$$\log \frac{1 - \varepsilon}{\varepsilon} \leq 2 \log \frac{1}{2\varepsilon}.$$

This proves (1.1) with  $4C_0 = 2^{19}$  for the entire range  $0 < \varepsilon < 1/2$ .  $\square$

To recover the additive formulation of [5, Conjecture 1.2], set

$$\delta = \frac{2^{19}}{(\log n)^2} \log \frac{1}{2\varepsilon},$$

and suppose  $\mu(p) > \varepsilon$ . Then  $p > p_\varepsilon$ , so Theorem 1.1 gives  $p + \delta > p_{1-\varepsilon}$ . Hence  $\mu(p + \delta) > 1 - \varepsilon$  whenever  $p + \delta \leq 1$ ; for larger increments, the endpoint 1 already has probability one. Conversely, when  $p_\varepsilon + \delta < 1$ , apply this strict formulation at  $p > p_\varepsilon$  and let  $p \downarrow p_\varepsilon$ . Continuity gives  $\mu(p_\varepsilon + \delta) \geq 1 - \varepsilon$ , which is the quantile bound. When  $p_\varepsilon + \delta \geq 1$ , that bound is automatic.

The same log-odds estimate describes the transition around its median  $p_* = p_{1/2}$ . The log odds vanish at  $p_*$ , so integration from the median gives, with  $C_0 = 2^{17}$ ,

$$\begin{aligned} \mu(p_* - s) &\leq \frac{1}{1 + \exp(s(\log n)^2/C_0)} \quad (0 \leq s \leq p_*), \\ 1 - \mu(p_* + s) &\leq \frac{1}{1 + \exp(s(\log n)^2/C_0)} \quad (0 \leq s \leq 1 - p_*). \end{aligned}$$

The endpoint cases follow by continuity.

## References

- [1] Aline Bonami, [Étude des coefficients de Fourier des fonctions de  \$L^p\(G\)\$](#) , *Annales de l'Institut Fourier* **20** (1970), no. 2, 335–402.
- [2] Jean Bourgain, Jeff Kahn, Gil Kalai, Yitzhak Katznelson, and Nathan Linial, [The influence of variables in product spaces](#), *Israel Journal of Mathematics* **77** (1992), 55–64.
- [3] Jean Bourgain and Gil Kalai, [Influences of variables and threshold intervals under group symmetries](#), *Geometric and Functional Analysis* **7** (1997), no. 3, 438–461.
- [4] Ehud Friedgut, [KKL's influence on me](#), in *Proceedings of the International Congress of Mathematicians 2022*, Vol. 6, Dmitry Beliaev and Stanislav Smirnov (eds.), EMS Press, 2023, pp. 4568–4581.
- [5] Ehud Friedgut and Gil Kalai, [Every monotone graph property has a sharp threshold](#), *Proceedings of the American Mathematical Society* **124** (1996), no. 10, 2993–3002.
- [6] Jeff Kahn, Gil Kalai, and Nathan Linial, [The influence of variables on Boolean functions](#), in *Proceedings of the 29th Annual Symposium on Foundations of Computer Science*, IEEE Computer Society, 1988, pp. 68–80.
- [7] Esty Kelman, Guy Kindler, Noam Lifshitz, Dor Minzer, and Muli Safra, [Towards a proof of the Fourier–entropy conjecture?](#), *Geometric and Functional Analysis* **30** (2020), no. 4, 1097–1138. Accessible preprint: [arXiv:1911.10579v2](#), 7 May 2020.
- [8] OpenAI, Integral and fractional expectation thresholds are equivalent, OpenAI Math Release preprint [OAI:Integral-and-fractional-expectation-thresholds-are-equivalent-September-23-2026](#), 2026.
- [9] OpenAI, The second Kahn–Kalai conjecture, OpenAI Math Release preprint [OAI:The-second-Kahn-Kalai-conjecture-September-24-2026](#), 2026.
- [10] Lucio Russo, [On the critical percolation probabilities](#), *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **56** (1981), no. 2, 229–237.