

A Smooth Nonquadratic Entire Affine Maximal Graph in Dimension Ten

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Abstract

We construct a smooth nonquadratic entire graph in dimension ten that solves the classical affine maximal equation and has positive-definite Hessian everywhere. This gives a smooth counterexample to the entire-graph affine Bernstein assertion in dimension ten. Hessian positivity is pointwise; no global uniform lower bound or completeness of the Berwald–Blaschke metric is asserted.

1 Introduction

For a smooth function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ with positive definite Hessian, write

$$\sum_{i,j=1}^n U^{ij} w_{ij} = 0, \quad U^{ij} = \det(D^2 u) (D^2 u)_{ij}^{-1}, \quad w = (\det D^2 u)^{-(n+1)/(n+2)}. \quad (1)$$

This is the classical affine maximal equation, the Euler–Lagrange equation of the affine area functional $\int (\det D^2 u)^{1/(n+2)}$ under compactly supported variations; see [5]. The entire-graph form of the affine Bernstein problem asks whether every such solution must be a quadratic polynomial. Throughout this paper, positivity of the Hessian means pointwise positive definiteness, not a global lower bound by a fixed positive matrix.

Remark (notation). u is a smooth real-valued graph function on its spatial domain, with pointwise positive definite Hessian. In Proposition 2.1: $u(x, t) = \sqrt{F(x)^2 + t^2}$ on $\Omega \times \mathbb{R}$, with $n = m + 1$. In Theorem 1.1: The constructed witness is $u(x, t) = \sqrt{F(|x|)^2 + t^2}$ for $(x, t) \in \mathbb{R}^9 \times \mathbb{R}$, using the global radial profile F . n is the positive integer number of independent variables of the graph function u . In Proposition 2.1: $n = m + 1$, where $m \geq 1$. In Theorem 1.1: $n = 10$. $D(r) = C(r)/B(r)$ is the normalized radial logarithmic coordinate, defined for $r > 0$ wherever $B(r) \neq 0$. In Lemma 4.2: In the vector-field and face calculations, D also denotes the freely varying third coordinate of $(A, B, D) \in \mathcal{B}$; this polynomial coordinate system is evaluated even on $B = 0$, without defining D there by division. i is a coordinate index whose range is specified by the indexed expression or summation. In Proposition 2.1: In X_i , e_i , F_i , F_{ij} , and P_{ij} , the spatial indices range from 1 to m ; in the full affine maximal equation they range from 1 to $n = m + 1$. In Theorem 1.1: In the displayed affine maximal equation, $1 \leq i, j \leq 10$. The notation j has a context-dependent meaning: a coordinate index in indexed tensor expressions and a flux function in the local integral construction. In Proposition 2.1, Theorem 1.1: j is a coordinate summation index with $1 \leq j \leq n$; in the ten-dimensional equation its range is $\{1, \dots, 10\}$. In Lemma 3.1: $j(s) = \int_0^1 v^{m-1} p(v^2 s)^{1-\gamma} dv$ for the current trial pair; at the fixed point, $j(r^2/2) = -y(r)^{m-1} P'(r)/(kr^m)$ for $r > 0$.

Theorem 1.1. *There exists a nonquadratic function $u \in C^\infty(\mathbb{R}^{10})$ with D^2u positive definite at every point such that*

$$\sum_{i,j=1}^{10} U^{ij} \partial_{ij} (\det D^2u)^{-11/12} = 0 \quad \text{on } \mathbb{R}^{10}, \quad U = \det(D^2u)(D^2u)^{-1}.$$

Remark (notation). $U = \det(D^2u)(D^2u)^{-1}$ is the cofactor matrix of the Hessian with respect to all n independent variables of u . In Proposition 2.1: $n = m + 1$, and U is evaluated on $\Omega \times \mathbb{R}$. In Theorem 1.1: $n = 10$, and U is evaluated on $\mathbb{R}^{10}$.

Thus the smooth entire-graph assertion has a negative answer in dimension ten. No growth assumption or completeness assumption for the Berwald–Blaschke metric is imposed here.

The classical problem has several formulations involving different notions of completeness. Chern formulated the two-dimensional entire-graph problem [2]. Calabi proved rigidity in dimension two assuming both Euclidean completeness and completeness of the affine metric [1]. Trudinger and Wang later proved that affine completeness implies Euclidean completeness for locally uniformly convex hypersurfaces of dimension at least two [6]. For the entire-graph problem, Trudinger and Wang proved that smooth locally uniformly convex entire affine maximal graphs over $\mathbb{R}^2$ are elliptic paraboloids [5]. In the same paper, they exhibited the singular dimension-ten example

$$u_0(x, t) = \sqrt{|x|^9 + t^2}, \quad (x, t) \in \mathbb{R}^9 \times \mathbb{R}, \quad (2)$$

see [5, Section 7]. This example is not smooth at the origin and does not have positive definite Hessian everywhere. Theorem 1.1 supplies a smooth example satisfying precisely these stronger pointwise requirements. The global section-based uniform convexity hypothesis in [5, Corollary 4.3] is stronger than the Hessian positivity assumed here.

Related equations with $w = (\det D^2u)^{-\theta}$ admit nonquadratic Euclidean-complete examples for other ranges of θ [3, 4]. The latter range extends to $\theta = n/(n + 1)$, but does not include the classical value $\theta = 11/12$ when $n = 10$.

Our construction keeps the relation $u^2 - t^2 = F(x)^2$ suggested by (2), but replaces $F(x) = |x|^{9/2}$ by a smooth strictly convex radial function with $F(0) = 1$. Section 2 gives a dimension-independent reduction of this ansatz to two second-order equations for positive functions F and P . Section 3 constructs their radial solution smoothly through the origin. The main obstacle is then global existence: a priori the auxiliary function P could vanish at a finite radius. Section 4 rules this out using a bounded invariant region for logarithmic derivatives, and completes the proof. The decisive change of derivative variables makes each component of the resulting vector field nondecreasing in the other two variables on a suitable box. A single stationary corner then controls all of its upper faces.

Remark (notation). F denotes the positive smooth function used in the graph ansatz, with the same letter also used for its radial profile when radial coordinates are adopted. In Proposition 2.1: F is a spatial function $F : \Omega \rightarrow (0, \infty)$ in dimension m , and derivatives $D_x^2 F$ are spatial derivatives. In Lemma 3.1, Lemma 4.1, Lemma 4.2, Proposition 4.3, Theorem 1.1: $F(r)$ denotes the radial profile in dimension $m = 9$, and the associated spatial function is $x \mapsto F(|x|)$; primes on F denote radial derivatives. P is the positive smooth auxiliary spatial function associated with F , with $P = (F / \det D_x^2 F)^{(m+2)/(m+3)}$. In Proposition 2.1: P is a function on the arbitrary open set $\Omega \subset \mathbb{R}^m$. In Lemma 3.1: P denotes a radial profile on $0 < r < \varepsilon$, normalized by $P(0) = 1$, whose spatial extension is $x \mapsto P(|x|)$ for $x \in \mathbb{R}^9$ near the origin. In Lemma 4.1, Lemma 4.2: P denotes the radial profile from the local construction, continued on the maximal outward interval on which $F, P, y > 0$, with spatial extension $x \mapsto P(|x|)$ in nine variables. In Proposition 4.3, Theorem 1.1:

P denotes the continued radial profile from the local construction; its global spatial extension is $x \mapsto P(|x|)$ on $\mathbb{R}^9$.

2 Reduction to a radial system

We first isolate the calculation that turns the fourth-order equation into a system for two functions of one fewer variable. For this section let $m \geq 1$, so that the graph has $m+1$ independent variables $(x, t) \in \mathbb{R}^m \times \mathbb{R}$. Set

$$k = m + 2, \quad \gamma = \frac{m+3}{m+2}.$$

We introduce P so that, for the ansatz below, the determinant power w separates into $P(x)$ times a fixed function of $t/F(x)$.

Remark (notation). $m \geq 1$ is an integer giving the number of x variables, so that (x, t) has $m+1$ independent variables. In Lemma 3.1, Lemma 4.1, Proposition 4.3, Theorem 1.1: $m = 9$. $k = m+2$ is the dimension-dependent scalar parameter. In Lemma 3.1, Lemma 4.1, Proposition 4.3, Theorem 1.1: $k = 11$. $\gamma = (m+3)/(m+2)$ is a fixed exponent determined by the spatial dimension $m \geq 1$. In Lemma 3.1, Lemma 4.1, Proposition 4.3, Theorem 1.1: $m = 9$ and $\gamma = 12/11$.

Proposition 2.1. *Let F, P be positive smooth functions on an open set $\Omega \subset \mathbb{R}^m$, and suppose $H = D_x^2 F$ is positive definite. If*

$$\det H = FP^{-\gamma}, \quad \text{tr}(H^{-1}D_x^2 P) + kP/F = 0, \quad (3)$$

then

$$u(x, t) = \sqrt{F(x)^2 + t^2}$$

has positive definite Hessian on $\Omega \times \mathbb{R}$ and satisfies (1) in dimension $n = m + 1$.

Remark (notation). $H(x) = D_x^2 F(x)$ is the $m \times m$ spatial Hessian on Ω , positive definite at every $x \in \Omega$.

Proof. Direct differentiation, interpreted as an identity of quadratic forms, gives

$$D^2 u = \frac{F}{u} H + \frac{F^2}{u^3} \left(dt - \frac{t}{F} dF \right)^2. \quad (4)$$

The first term is positive on the x directions, and the second is positive on the remaining direction. Thus $D^2 u > 0$. Put $\zeta = t/F$ and use the pointwise basis

$$X_i = e_i + \zeta F_i e_t \quad (1 \leq i \leq m), \quad e_t.$$

Its change-of-basis matrix has determinant one. In this basis, (4) has diagonal blocks $(F/u)H$ and F^2/u^3 , so

$$\det D^2 u = F^{m+2} u^{-(m+3)} \det H. \quad (5)$$

The first equation in (3) is equivalent to $P = (F/\det H)^{(m+2)/(m+3)}$. Therefore

$$w = (\det D^2 u)^{-(m+2)/(m+3)} = Ph(\zeta), \quad h(\zeta) = (1 + \zeta^2)^{k/2}. \quad (6)$$

Remark (notation). $\zeta(x, t) = t/F(x)$ is a smooth real-valued function on $\Omega \times \mathbb{R}$. e_i is the standard coordinate vector in the i th x direction of $\mathbb{R}^m \times \mathbb{R}$, with $1 \leq i \leq m$. For $(x, t) \in \Omega \times \mathbb{R}$ and

$1 \leq i \leq m$, $X_i(x, t) = e_i + \zeta(x, t)F_i(x)e_t$, where $F_i = \partial_i F$; together with e_t , these vectors form a pointwise basis.

We evaluate the ordinary Hessian of w in the same pointwise basis. Since $d\zeta(X_i) = 0$, twice differentiating $t = F\zeta$ gives

$$D^2\zeta(X_i, X_j) = -\frac{\zeta}{F}F_{ij}.$$

Consequently

$$D^2w(X_i, X_j) = hP_{ij} - \frac{P}{F}\zeta h'F_{ij}, \quad w_{tt} = \frac{P}{F^2}h''.$$

Taking the trace against the inverse blocks of D^2u yields

$$\frac{F}{u} \operatorname{tr}((D^2u)^{-1}D^2w) = h \operatorname{tr}(H^{-1}D_x^2P) + \frac{P}{F}((1 + \zeta^2)h'' - m\zeta h'). \quad (7)$$

Differentiation of h gives

$$(1 + \zeta^2)h'' - m\zeta h' = k(1 + \zeta^2)^{k/2-1}(1 + (k-1-m)\zeta^2) = kh,$$

because $k = m + 2$. The second equation in (3) makes (7) zero. Multiplication by the positive determinant in (5) gives (1). $\square$

Remark (notation). $d = |z - \Pi z|$ is the nonnegative Euclidean distance from z to $\mathcal{B}$. In Lemma 4.2: Along a trajectory z , d depends on $\log r$ through z , and the differentiated quantity is d^2 .

We now take F and P radial, and use the same letters for their profiles in $r = |x|$. Write primes for radial derivatives and $y = F'$. For $r > 0$ the eigenvalues of D_x^2F are y' in the radial direction and y/r in the $m-1$ tangential directions. Thus, whenever $y, y' > 0$,

$$\det D_x^2F = y' \left(\frac{y}{r}\right)^{m-1}, \quad \operatorname{tr}((D_x^2F)^{-1}D_x^2P) = \frac{P''}{y'} + (m-1)\frac{P'}{y}.$$

Substituting into (3), and multiplying the second equation by $y'y^{m-1}$, gives the equivalent system

$$y^{m-1}y' = r^{m-1}FP^{-\gamma}, \quad (y^{m-1}P')' = -kr^{m-1}P^{1-\gamma}, \quad F' = y. \quad (8)$$

The divergence form of the second equation will allow us to start the solution at the singular point $r = 0$.

Remark (notation). $y(r) = F'(r)$ denotes the derivative of the radial profile with respect to r , rather than a spatial partial derivative of $F(x)$.

3 A smooth radial solution at the origin

From now on fix

$$m = 9, \quad k = 11, \quad \gamma = 12/11. \quad (9)$$

We construct a local solution with $F(0) = P(0) = 1$. The radial equations contain divisions by $y = F'$, which vanishes at the origin. An integral formulation in $s = r^2/2$ avoids this singularity and makes smoothness in x explicit.

Remark (notation). s ranges over the auxiliary interval $[-\delta, \delta]$ in the integral construction; its physical radial values are $s = r^2/2 \geq 0$.

Lemma 3.1. *There is $\varepsilon > 0$ and a solution F, P of (8) on $0 < r < \varepsilon$ such that $F(|x|)$ and $P(|x|)$ extend smoothly to $|x| < \varepsilon$, with*

$$F(0) = P(0) = 1, \quad D_x^2 F(0) = I, \quad D_x^2 P(0) = -\frac{11}{9}I.$$

Moreover $F, P, y, y' > 0$ and $P' < 0$ for $0 < r < \varepsilon$.

Proof. Seek $F(r) = f(s)$ and $P(r) = p(s)$, where $s = r^2/2$. The integrated equations suggest the normalized derivative $q = y/r$ and flux $j = -y^{m-1}P'/(kr^m)$ for $r > 0$. We will construct their regular extensions as functions of s using the following averages. For continuous f, p on $[-\delta, \delta]$ taking values in $[1/2, 3/2]$, define

$$q(s) = \left(m \int_0^1 v^{m-1} f(v^2 s) p(v^2 s)^{-\gamma} dv \right)^{1/m}, \quad (10)$$

$$j(s) = \int_0^1 v^{m-1} p(v^2 s)^{1-\gamma} dv. \quad (11)$$

On the closed subset

$$\mathcal{K} = \{(f, p) \in C([-\delta, \delta])^2 : \|f - 1\|_\infty, \|p - 1\|_\infty \leq 1/2\},$$

consider

$$\mathcal{T}(f, p)(s) = \left(1 + \int_0^s q(a) da, 1 - k \int_0^s j(a) q(a)^{1-m} da \right). \quad (12)$$

The functions q, j have uniform positive upper and lower bounds. The maps $(f, p) \mapsto q$ and $(f, p) \mapsto j q^{1-m}$ are Lipschitz in the product supremum norm, with constants independent of δ : all scalar powers are evaluated on fixed positive compact intervals. Hence there are constants M, L independent of δ such that

$$\|\mathcal{T}(f, p) - (1, 1)\|_\infty \leq M\delta, \quad \|\mathcal{T}(f, p) - \mathcal{T}(\tilde{f}, \tilde{p})\|_\infty \leq L\delta\|(f, p) - (\tilde{f}, \tilde{p})\|_\infty.$$

Choose $M\delta \leq 1/2$ and $L\delta < 1$. The contraction mapping theorem gives a fixed point in $\mathcal{K}$.

Remark (notation). $\delta > 0$ is the half-length of the auxiliary interval $[-\delta, \delta]$, chosen sufficiently small that $M\delta \leq 1/2$ and $L\delta < 1$. f is the first component of a trial pair $(f, p) \in \mathcal{K}$; when the pair is the constructed fixed point, it supplies the radial profile through $F(r) = f(r^2/2)$ and is smooth on $(-\delta, \delta)$. p is the second component of a trial pair $(f, p) \in \mathcal{K}$; when the pair is the constructed fixed point, it supplies the radial profile through $P(r) = p(r^2/2)$ and is smooth on $(-\delta, \delta)$. q is defined by the positive m th root in (10) for each trial pair $(f, p) \in \mathcal{K}$ on $[-\delta, \delta]$; at the constructed fixed point it is smooth on $(-\delta, \delta)$ and satisfies $q(r^2/2) = y(r)/r$ for $r > 0$. M is a constant independent of δ and of the choice of $(f, p) \in \mathcal{K}$, such that $\|\mathcal{T}(f, p) - (1, 1)\|_\infty \leq M\delta$ uniformly on $\mathcal{K}$. L denotes a Lipschitz bound local to the construction in which it occurs. In Lemma 3.1: L is independent of δ and satisfies $\|\mathcal{T}(f, p) - \mathcal{T}(\tilde{f}, \tilde{p})\|_\infty \leq L\delta\|(f, p) - (\tilde{f}, \tilde{p})\|_\infty$. In Lemma 4.2: L is a Lipschitz constant for V on the chosen bounded neighborhood of the box.

The fixed point is smooth on $(-\delta, \delta)$. Indeed, continuity of q, j first gives $f, p \in C^1$. If f, p are C^ℓ , differentiation under the integrals in (10)–(11) shows that q, j are C^ℓ , and (12) gives one further derivative. This proves the assertion by induction. The definition on both sides of $s = 0$ ensures that $f(|x|^2/2)$ and $p(|x|^2/2)$ are smooth in x at the origin.

Remark (notation). $C(r) = -rP'(r)/P(r)$ denotes the radial logarithmic coordinate for $r > 0$ wherever $P(r) \neq 0$, with the prime denoting differentiation with respect to r .

For $r > 0$, the fixed point equations give $y = rq(r^2/2) > 0$. Changing variables $\rho = rv$ in (10)–(11) then yields

$$y^m = m \int_0^r \rho^{m-1} F(\rho) P(\rho)^{-\gamma} d\rho, \quad y^{m-1} P' = -k \int_0^r \rho^{m-1} P(\rho)^{1-\gamma} d\rho. \quad (13)$$

Differentiation proves (8); the first equation implies $y' > 0$, and the second integral gives $P' < 0$. Finally $q(0) = 1$ and $j(0) = 1/m$, so

$$f_s(0) = 1, \quad p_s(0) = -k/m = -11/9.$$

These are exactly the claimed Hessians of the radial extensions. Shrinking ε if necessary completes the proof. $\square$

4 An invariant box and global continuation

The local solution now has every required property. It remains to show that it can be continued to arbitrarily large radii without losing positivity. We do this by controlling logarithmic derivatives, rather than by estimating F and P separately.

Take the maximal outward continuation of the local solution on which $F, P, y > 0$. This is a regular smooth ODE for $r > 0$. Explicitly, with $Z = P'$ and

$$G(r, F, P, y) = \frac{r^8 F P^{-12/11}}{y^8},$$

it reads

$$F' = y, \quad P' = Z, \quad y' = G, \quad Z' = -8 \frac{G}{y} Z - \frac{11 r^8 P^{-1/11}}{y^8}. \quad (14)$$

In particular $y' > 0$ throughout this continuation. Define

$$A = \frac{ry}{F}, \quad B = \frac{ry'}{y}, \quad C = -\frac{rP'}{P}, \quad (15)$$

and let a dot mean differentiation with respect to $\log r$. Since $B > 0$, we may also set $D = C/B$. This normalization gives a vector field with nonnegative cross derivatives on the box in Lemma 4.2, allowing its upper faces to be controlled by one corner.

Remark (notation). $B(r) = ry'(r)/y(r)$ is the second logarithmic coordinate of a radial profile, defined for $r > 0$ wherever $y(r) \neq 0$, with the prime denoting differentiation with respect to r . In Lemma 4.2: In the vector-field and face calculations, B also denotes the freely varying second coordinate of $(A, B, D) \in \mathcal{B}$, including the face $B = 0$. $Z(r) = P'(r)$ is a smooth real-valued function on the maximal outward continuation of the local solution for $r > 0$ on which $F, P, y > 0$, with the prime denoting differentiation with respect to r . $G(r, F, P, y) = r^8 F P^{-12/11} / y^8$ is defined on the domain $r, F, P, y > 0$. $A(r) = ry(r)/F(r)$ is the first logarithmic coordinate of a radial profile, defined for $r > 0$ wherever the denominator is nonzero, with primes denoting radial derivatives. In Lemma 4.2: In the vector-field and face calculations, A also denotes the freely varying first coordinate of $(A, B, D) \in \mathcal{B}$, rather than only its value along the constructed solution.

Lemma 4.1. *On the interval of continuation, $A, B > 0$, and the functions A, B, D satisfy*

$$\begin{aligned} \dot{A} &= A(1 + B - A), \\ \dot{B} &= B(9 + A + (12/11)BD - 9B), \\ \dot{D} &= (11 - D)A - 8D + BD(1 - D/11). \end{aligned} \quad (16)$$

Moreover $(A, B, D) \rightarrow (0, 1, 0)$ as $r \downarrow 0$, and all three coordinates are positive for sufficiently small $r > 0$.

Proof. Positivity of F, y, y' gives $A, B > 0$. Using (8), one may write

$$B = \frac{r^m F P^{-\gamma}}{y^m}.$$

Logarithmic differentiation of A and B , followed by the second equation in (8), gives

$$\begin{aligned}\dot{A} &= A(1 + B - A), \\ \dot{B} &= B(m + A + \gamma C - mB), \\ \dot{C} &= C(1 + C - (m - 1)B) + kAB.\end{aligned}$$

For the last identity, use $\dot{C} = C + C^2 - r^2 P''/P$ and $AB = r^{m+1} P^{-\gamma}/y^{m-1}$. Thus

$$\dot{D} = (k - D)A - (m - 1)D + BD(1 + (1 - \gamma)D).$$

Substituting (9) proves (16).

Lemma 3.1 gives $F = 1 + O(r^2)$, $y = r + O(r^3)$, $y' = 1 + O(r^2)$, and $P' = -(11/9)r + O(r^3)$. Hence $A \rightarrow 0$, $B \rightarrow 1$, and $D \rightarrow 0$. The positivity of A, B and the local inequality $P' < 0$ also give $D > 0$ for small positive r . $\square$

The singular profile (2) suggests the bounds for these variables. Its radial factor $F = r^{9/2}$ has $A = 9/2$ and $B = 7/2$; the associated function $P = (F/\det D_x^2 F)^{11/12}$ is a positive constant times $r^{-33/2}$, giving $C = 33/2$ and $D = 33/7$. These constants determine an invariant box for the smooth solution.

Lemma 4.2. *The box*

$$\mathcal{B} = [0, 9/2] \times [0, 7/2] \times [0, 33/7] \tag{17}$$

is forward invariant for (16). The solution arising from Lemma 3.1 lies in $\mathcal{B}$ throughout its interval of continuation.

Proof. Write $V = (V_1, V_2, V_3)$ for the vector field in (16). The upper corner $a_* = (9/2, 7/2, 33/7)$ satisfies $V(a_*) = 0$ by direct substitution. Each V_i is nondecreasing in the other two coordinates on $\mathcal{B}$. The nonzero cross derivatives are

$$\partial_B V_1 = A, \quad \partial_A V_2 = B, \quad \partial_D V_2 = (12/11)B^2, \quad \partial_A V_3 = 11 - D, \quad \partial_B V_3 = D(1 - D/11),$$

all nonnegative because $0 \leq D \leq 33/7 < 11$. On each upper face, comparison with a_* therefore gives the corresponding $V_i \leq 0$. On the lower faces $A = 0$, $B = 0$, and $D = 0$, the respective components are 0, 0, and $11A$, hence nonnegative.

These weak boundary inequalities suffice for invariance. Let Πz be the coordinatewise projection of $z \in \mathbb{R}^3$ onto $\mathcal{B}$. The face inequalities imply

$$(z - \Pi z) \cdot V(\Pi z) \leq 0.$$

On any bounded neighborhood of the box, V has a Lipschitz constant L . Along a trajectory z in that neighborhood, the squared distance $d^2 = |z - \Pi z|^2$ consequently satisfies

$$\frac{d}{d \log r} d^2 = 2(z - \Pi z) \cdot V(z) \leq 2Ld^2.$$

The squared distance to a closed box is continuously differentiable, so this identity also holds when the nearest face changes. Gronwall's inequality shows that a trajectory starting in the box cannot leave it.

Remark (notation). z denotes a point of $\mathbb{R}^3$, with coordinates corresponding to (A, B, D) . In Lemma 4.2: In the trajectory-distance argument, $z = z(\log r)$ satisfies $\dot{z} = V(z)$ while lying in the bounded neighborhood under consideration.

Finally, Lemma 4.1 gives strictly positive coordinates tending to $(0, 1, 0)$ as $r \downarrow 0$. They therefore lie below all three upper bounds for sufficiently small $r > 0$. Forward invariance proves the last assertion. $\square$

Proposition 4.3. *The solution in Lemma 3.1 extends to every $r > 0$, with $F, P, y, y' > 0$. Its radial extensions to $\mathbb{R}^9$ are smooth, and $D_x^2 F$ is positive definite everywhere.*

Proof. Suppose its maximal outer endpoint were $R < \infty$, and fix $r_0 \in (0, R)$. Lemma 4.2 gives

$$0 \leq A \leq 9/2, \quad 0 \leq B \leq 7/2, \quad 0 \leq C = BD \leq 33/2.$$

Since the derivatives of $\log F, \log y, \log P$ with respect to $\log r$ are $A, B, -C$, respectively, integration gives, for $r_0 \leq r < R$,

$$\begin{aligned} F(r_0) &\leq F(r) \leq F(r_0)(R/r_0)^{9/2}, \\ y(r_0) &\leq y(r) \leq y(r_0)(R/r_0)^{7/2}, \\ P(r_0)(r_0/R)^{33/2} &\leq P(r) \leq P(r_0). \end{aligned}$$

Also $|P'(r)| = C(r)P(r)/r \leq (33/2)P(r_0)/r_0$. Thus (r, F, P, y, Z) remains in a compact subset of the domain $r, F, P, y > 0$ of the smooth system (14). Its right-hand side is bounded there, so the state has a limit at R in that domain. Local existence and uniqueness at this limit extend the solution past R , a contradiction.

Smoothness for $r > 0$ follows from (14), and Lemma 3.1 supplies smoothness in x at the origin. The Hessian eigenvalues for $r > 0$ are $y' > 0$ and $y/r > 0$; at the origin the Hessian is I . $\square$

Proof of Theorem 1.1. Take the global radial functions of Proposition 4.3. By (8), they satisfy (3) for $x \neq 0$. The functions are smooth, $F, P > 0$, and $D_x^2 F > 0$ everywhere, so both identities extend to $x = 0$ by continuity. Proposition 2.1 now shows that

$$u(x, t) = \sqrt{F(|x|)^2 + t^2}, \quad (x, t) \in \mathbb{R}^9 \times \mathbb{R},$$

is smooth, has positive definite Hessian everywhere, and solves the classical affine maximal equation in dimension ten. Finally, $u(0, t) = \sqrt{1 + t^2}$ because $F(0) = 1$, so u is not a quadratic polynomial. $\square$

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