

A uniform influence bound for hypergraph properties

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Abstract

For every fixed integer $r \geq 3$, we prove that every relabeling-invariant Boolean property of simple r -uniform hypergraphs on n vertices satisfies $\text{Var}_p(f) \leq C_r I_p(f) / (\log n)^{r/(r-1)}$. The constant depends only on r , and the bound holds uniformly for all $0 < p < 1$ without a monotonicity assumption. For increasing properties, it gives the corresponding threshold-width bound with exponent $r/(r-1)$, proving the hypergraph threshold-width conjecture of Friedgut and Kalai.

1 Introduction

Let $r \geq 3$ and $n \geq r$ be integers, write $[n] = \{1, \dots, n\}$, and put $E_{n,r} = \binom{[n]}{r}$. An element of $\{0, 1\}^{E_{n,r}}$ represents a simple r -uniform hypergraph. A *hypergraph property* is a Boolean function on this space that is invariant under every permutation of $[n]$. For $0 < p < 1$, give the coordinates the independent Bernoulli(p) law, and denote expectation and variance by $\mathbb{E}_p$ and Var_p .

Remark (notation). p denotes the common Bernoulli bias, with $p \in [0, 1]$. In Theorem 1.1, Lemma 2.1, Lemma 3.1: $0 < p < 1$. In Corollary 4.1: In $q(p) = \mathbb{E}_p f$ and the definition of p_a , p ranges over $[0, 1]$, including the endpoint product laws. A is a subset of the current coordinate set indexing a Fourier basis function or coefficient. In Lemma 2.1: In the difference identity $\widehat{h}(A \cup \{i\}) = \sigma \widehat{\Delta_i h}(A)$, $A \subseteq J \setminus \{i\}$. In Lemma 3.1: In the expansion of $\widehat{f_y}(A)$, $A \subseteq F_B$.

For an edge e , let $x_{e \leftarrow b}$ be x with its e -coordinate set to b . Define the *pivotal influence* of e and the *total influence* of f by

$$I_{e,p}(f) = \Pr_p(f(x_{e \leftarrow 1}) \neq f(x_{e \leftarrow 0})), \quad I_p(f) = \sum_{e \in E_{n,r}} I_{e,p}(f).$$

The probability in the first expression is over the other coordinates; it does not include an additional factor of $p(1-p)$. All logarithms are natural, and n always counts vertices.

Remark (notation). x denotes a configuration in $\{0, 1\}^J$ for the coordinate set on which the function being evaluated is defined. In Theorem 1.1, Corollary 4.1: For evaluations of f , the coordinate set is $E_{n,r}$. In Lemma 3.1: The coordinate set is $E = E_{n,r}$ for f , and F_B for the restricted function f_y . b is locally bound, with its domain determined by the expression in which it occurs. In Theorem 1.1, Lemma 3.1, Corollary 4.1: In the coordinate replacement notation $x_{e \leftarrow b}$, $b \in \{0, 1\}$ is the replacement value. In Lemma 2.1: In the one-coordinate norm calculation, $b \in \mathbb{R}$ is an arbitrary scalar. In the product induction, $g = a + b\chi_i$ uses a real-valued coefficient function b on $\{0, 1\}^{J \setminus \{i\}}$.

Theorem 1.1. *For every integer $r \geq 3$ there is a finite constant C_r such that, for every integer $n \geq r$, every $0 < p < 1$, and every hypergraph property $f : \{0, 1\}^{E_{n,r}} \rightarrow \{0, 1\}$,*

$$\mathrm{Var}_p(f) \leq \frac{C_r}{(\log n)^{r/(r-1)}} I_p(f). \quad (1.1)$$

No monotonicity assumption is required.

The theorem measures how vertex symmetry forces sensitivity to edge changes. If the probability of a property stays bounded away from zero and one, its total influence must grow at least as $(\log n)^{r/(r-1)}$.

A property is *increasing* if changing coordinates from 0 to 1 cannot decrease its value, and *nontrivial* if it is not constant. For a nontrivial increasing property, put $q(p) = \mathbb{E}_p f$ and define $p_a = \inf\{p \in [0, 1] : q(p) \geq a\}$ for $0 < a < 1$. For fixed $0 < \varepsilon < 1/2$, its *threshold width* is $p_{1-\varepsilon} - p_\varepsilon$. Friedgut and Kalai conjectured that this width is at most $O_{r,\varepsilon}((\log n)^{-r/(r-1)})$ for every increasing r -uniform hypergraph property [5, Section 5, p. 2999]. Their formulation counts edge coordinates rather than vertices; for fixed r , $\log \binom{n}{r} \sim r \log n$ as $n \rightarrow \infty$. Corollary 4.1 proves this conjectured bound by integrating (1.1), with an explicit dependence on ε .

Remark (notation). The letter a is locally bound, with its domain and meaning determined by the calculation in which it occurs. In Corollary 4.1: $a \in (0, 1)$ is the probability level indexing p_a . In Lemma 2.1: In the one-coordinate calculation, $a \in \mathbb{R}$ is a coefficient; in the product induction $g = a + b\chi_i$, it is a real-valued function on $\{0, 1\}^{J \setminus \{i\}}$.

Context and attribution. Kahn, Kalai, and Linial [6] established a fundamental link between Fourier analysis and the influences of Boolean functions. For a function invariant under a transitive group on M coordinates, their theorem gives a total-influence lower bound of order $\mathrm{Var}_{1/2}(f) \log M$ at the uniform measure. Bourgain, Kahn, Kalai, Katznelson, and Linial [3] extended the influence method to product spaces. Friedgut and Kalai [5] used influence inequalities to prove sharp thresholds for increasing families with transitive symmetry. For hypergraph properties, the induced action on the $\binom{n}{r}$ edge coordinates is transitive. The richer structure of vertex permutations, however, contains more information than this single transitivity statement.

Bourgain and Kalai [4, Corollary 4.1] studied how group actions on sets of coordinates improve influence estimates. For fixed r and every $\eta > 0$, their result gives a lower bound of order $\mathrm{Var}_{1/2}(f)(\log n)^{r/(r-1)-\eta}$ for the total influence of a hypergraph property at the uniform measure. Their influence results already allow nonmonotone functions. Kelman, Kindler, Lifshitz, Minzer, and Safra obtained further estimates for group-invariant functions [7]. For graph properties, these give $I_{1/2}(f) \geq c \mathrm{Var}_{1/2}(f)(\log n)^2 / (\log \log n)^2$ for sufficiently large n ; see the arXiv version cited in [7, Section 7.4]. The present proof uses vertex symmetry through restrictions: all edges touching a vertex block remain free, and only edges wholly outside the block are fixed. This is the method of the companion manuscript *A Sharp Threshold Bound for Monotone Graph Properties* [9, Theorem 1.2 and Sections 2–3], which establishes the corresponding variance bound with denominator $(\log n)^2$ for graph properties, uniformly in p and without monotonicity. We reproduce its Fourier estimate and restriction argument in full. The additional combinatorial step replaces the graph degree estimate by its r -uniform counterpart, producing the exponent $r/(r-1)$.

Proof strategy. Put $\sigma = \sqrt{p(1-p)}$ and $\alpha = r/(r-1)$. The Fourier expansion for the Bernoulli product measure is indexed by sets S of hyperedges. Its squared nonconstant coefficients sum to the variance; weighting them by $|S|$ gives $\sigma^2 I_p(f)$. Thus the influence controls high degrees. The task is to bound the remaining mass using symmetry.

Remark (notation). S denotes a subset of the current coordinate set, used as a Fourier index or, for hypergraph coordinates, as an edge support. In Lemma 2.1: $S \subseteq J$, and the low-degree sum ranges over $1 \leq |S| \leq k$. In Theorem 1.1, Lemma 3.1: $S \subseteq E_{n,r}$ indexes the Fourier expansion of f . In Lemma 3.2: $S \subseteq E_{n,r}$ is a fixed support with $1 \leq |S| \leq L$, considered before B is sampled.

For large n , choose a uniformly random block B of m vertices, with m about $\sqrt{n}$, and leave free all edges meeting B . Every permutation within B survives an assignment to the other coordinates, and every free-edge orbit has at least m elements. Section 2 bounds the restricted Fourier mass up to a degree k proportional to $\sigma \log m$ by $m^{-1/4}$ times the restricted total influence. Its linear dependence on influence allows averaging over the assigned coordinates.

Remark (notation). m is a fixed real parameter with $m \geq 1$. In Lemma 2.1: $m \geq 2$ is real and is the parameter in the individual-influence bound $I_i(h) \leq I(h)/m$ and the cutoff $k = \sigma \log m/64$. In Lemma 3.1: m is an integer with $r+1 \leq m \leq n$, and $|B| = m$. In Lemma 3.2: m is an integer with $1 \leq m \leq n$, and $|B| = m$. In Theorem 1.1: In the proof for $n \geq N_r$, $m = \lfloor \sqrt{n} \rfloor$ is the block size. k is a real degree cutoff; comparisons of integer cardinalities with k require no rounding. In Lemma 2.1: $k = \sigma \log m/64$ with real $m \geq 2$; the low-degree sum is empty if $k < 1$. The auxiliary projection inequalities are established for every real cutoff $k \geq 0$. In Lemma 3.1: $k = \sigma \log m/64$ with integer $r+1 \leq m \leq n$. In Lemma 3.2: $k > 0$ is arbitrary, and $L = (k/r)^{r/(r-1)}$. In Theorem 1.1: In the branch $n \geq N_r$, $m = \lfloor \sqrt{n} \rfloor$ and $k = \sigma \log m/64 > 0$.

Regard a nonempty Fourier index S as an r -uniform hypergraph with s edges. It has a vertex of positive degree at most $rs^{(r-1)/r}$. For $s \leq L = (k/r)^\alpha$, this degree is at most k . The event that B meets the union of these edges only at that vertex has probability comparable to m/n and leaves between one and k support edges free. After averaging over the assignments and B , the restricted total influence is at most $(rm/n)I_p(f)$, so the two factors of m/n cancel. Section 3 thereby controls the original Fourier mass up to degree L .

Remark (notation). $L = (k/r)^\alpha$ is a positive real support-size cutoff. In Lemma 3.2: $k > 0$ is an arbitrary cutoff, so L is not tied to $\sigma \log m/64$. In Theorem 1.1: In the branch $n \geq N_r$, $m = \lfloor \sqrt{n} \rfloor$, $k = \sigma \log m/64$, and hence $L = (\sigma \log m/(64r))^\alpha$. $s = |S|$ is the number of edges in the support under consideration. In Lemma 3.2: $1 \leq s \leq L$.

Since L is of order $(\sigma \log n)^\alpha$, the remaining mass is at most $\sigma^2 I_p(f)/L$, with coefficient of order $\sigma^{2-\alpha}(\log n)^{-\alpha}$. The positive exponent $2 - \alpha$ makes this bound uniform in p . Section 4 completes the estimate and derives the threshold consequence.

2 A Fourier estimate for small individual influences

We first work on an arbitrary finite coordinate set J . Our objective is to control low-degree Fourier mass when every individual influence is small relative to the total. We follow the elementary argument in [9, Lemma 2.1]; its analytic ingredient is a two-to-four norm estimate of the type underlying the classical hypercontractive inequalities [2, 1]. The proof below includes the dependence on the bias.

Remark (notation). J denotes the finite coordinate set of the product space currently under consideration. In Theorem 1.1, Lemma 3.1: For the Fourier expansion of f , $J = E = E_{n,r}$; when the finite-coordinate estimate is applied to a restricted function f_y , its coordinate set is $J = F_B$.

Fix $0 < p < 1$ and put $\sigma = \sqrt{p(1-p)}$. For $i \in J$ and $A \subseteq J$, define

$$\chi_i(x) = \frac{x_i - p}{\sigma}, \quad \chi_A(x) = \prod_{i \in A} \chi_i(x), \quad \chi_\emptyset = 1.$$

Independence, zero means, and unit second moments show that the χ_A form an orthonormal basis.

Write $\widehat{h}(A) = \mathbb{E}_p[h\chi_A]$. Throughout this section, norms and expectations use this product law, and $I_i(h)$ and $I(h)$ denote pivotal and total influence on J .

For Boolean h , let $\Delta_i h$ be the function on the other coordinates obtained by subtracting the section at $x_i = 0$ from the section at $x_i = 1$. Direct expectation in coordinate i gives

$$\widehat{h}(A \cup \{i\}) = \sigma \widehat{\Delta_i h}(A) \quad (A \subseteq J \setminus \{i\}).$$

Because $\Delta_i h \in \{-1, 0, 1\}$, Parseval's identity yields

$$\sum_{S \subseteq J} |S| \widehat{h}(S)^2 = \sigma^2 I(h), \quad \text{Var}_p(h) = \sum_{\emptyset \neq S \subseteq J} \widehat{h}(S)^2. \quad (2.1)$$

In particular, this identity uses no sign condition on $\Delta_i h$.

Lemma 2.1 (Low-degree mass). *Let $m \geq 2$ be real and let $h : \{0, 1\}^J \rightarrow \{0, 1\}$ satisfy $I_i(h) \leq I(h)/m$ for every $i \in J$. With $k = \sigma \log m / 64$, one has*

$$\sum_{1 \leq |S| \leq k} \widehat{h}(S)^2 \leq m^{-1/4} I(h). \quad (2.2)$$

The sum is empty when $k < 1$.

Proof. We will use a two-to-four norm estimate to bound low-degree projections of the differences $\Delta_i h$. Set $\rho = \sigma/4$ and define the linear operator T by $T\chi_A = \rho^{|A|}\chi_A$. We first prove

$$\|Tg\|_4 \leq \|g\|_2 \quad (2.3)$$

for every real-valued function g on the product space. For one coordinate, $|\chi_i| \leq 1/\sigma$ and $\mathbb{E}\chi_i^2 = 1$ imply $|\mathbb{E}\chi_i^3| \leq 1/\sigma$ and $\mathbb{E}\chi_i^4 \leq 1/\sigma^2$. For real a, b , expansion and $2|ab^3| \leq a^2b^2 + b^4$ give

$$\begin{aligned} \mathbb{E}(a + \rho b \chi_i)^4 &\leq a^4 + 6\rho^2 a^2 b^2 + \frac{2\rho^3}{\sigma}(a^2 b^2 + b^4) + \frac{\rho^4}{\sigma^2} b^4 \\ &= a^4 + \frac{13\sigma^2}{32} a^2 b^2 + \frac{9\sigma^2}{256} b^4 \leq (a^2 + b^2)^2, \end{aligned}$$

where the last inequality uses $\sigma \leq 1/2$.

Remark (notation). On any current coordinate set $K \subseteq J$, T denotes the linear operator on real-valued functions on $\{0, 1\}^K$ determined by $T\chi_A = \rho^{|A|}\chi_A$ for every $A \subseteq K$, with the same p and ρ .

To extend this estimate to the product, induct on $|J|$, starting with the constant functions when J is empty. Write $g = a + b\chi_i$, where a, b depend on the remaining coordinates, and use T for the operator on those coordinates as well. The one-coordinate estimate, applied pointwise, followed by the L^2 triangle inequality gives

$$\|Tg\|_4^2 \leq \|(Ta)^2 + (Tb)^2\|_2 \leq \|Ta\|_4^2 + \|Tb\|_4^2 \leq \|a\|_2^2 + \|b\|_2^2 = \|g\|_2^2.$$

This proves (2.3).

Let $P_{\leq k}$ be the orthogonal projection onto the span of χ_A with $|A| \leq k$. For every u in this span,

$$\|u\|_4 \leq \|T^{-1}u\|_2 \leq \rho^{-k} \|u\|_2.$$

Taking the supremum of $|\mathbb{E}[gu]|$ over unit L^2 vectors in this span, and applying Hölder's inequality, therefore gives

$$\|P_{\leq k}g\|_2 \leq \rho^{-k}\|g\|_{4/3}. \quad (2.4)$$

These inequalities are valid for real $k \geq 0$, with no rounding required.

Remark (notation). u is a locally bound auxiliary object whose domain is specified in the argument using it. In Lemma 2.1: $u \in \text{span}\{\chi_A : A \subseteq J, |A| \leq k\}$ is a real-valued function; in the dual characterization of the projection norm, the supremum is over such functions with $\|u\|_2 = 1$. In Lemma 3.2: $u \in V = \bigcup_{e \in S} e$ is a vertex chosen with $1 \leq d_S(u) \leq k$; this choice depends only on S and is made before B is sampled.

Each nonempty Fourier support is counted at least once when summed over its coordinates. Using the difference identity and enlarging the resulting degree cutoff from $k-1$ to k , we obtain

$$\begin{aligned} \sum_{1 \leq |S| \leq k} \widehat{h}(S)^2 &\leq \sigma^2 \sum_{i \in J} \|P_{\leq k} \Delta_i h\|_2^2 \\ &\leq \sigma^2 \rho^{-2k} \sum_{i \in J} I_i(h)^{3/2}. \end{aligned} \quad (2.5)$$

Here each projection acts on the coordinates other than i , and $\|\Delta_i h\|_{4/3}^2 = I_i(h)^{3/2}$ because $|\Delta_i h|$ is an indicator.

The function $\sigma \log(4/\sigma)$ is increasing on $(0, 1/2]$ and is at most 2. Hence $\rho^{-2k} \leq m^{1/16}$. If $I(h) \leq m^{1/4}$, then $I_i(h)^{1/2} \leq m^{-3/8}$, so (2.5) is at most $\sigma^2 m^{-5/16} I(h) \leq m^{-1/4} I(h)$. If $I(h) > m^{1/4}$, the right side of (2.2) exceeds 1, whereas its left side is at most $\text{Var}_p(h) \leq 1/4$. $\square$

3 Restriction to a vertex block

Return to a hypergraph property f on $E = E_{n,r}$ and fix the bias p , with $\sigma = \sqrt{p(1-p)}$. Its Fourier coefficients use the basis of Section 2, with $J = E$. The restriction method of [9, Lemma 3.1 and Proposition 3.2] preserves a large permutation group after conditioning. We first obtain an averaged estimate for restricted Fourier degrees, and then relate those degrees to the sizes of the original supports.

For a vertex set $B \subseteq [n]$, write

$$F_B = \{e \in E : e \cap B \neq \emptyset\}$$

for the edge coordinates that remain free.

Lemma 3.1 (Averaged restriction estimate). *Let $r+1 \leq m \leq n$ be integers and set $k = \sigma \log m/64$. For B chosen uniformly among the m -subsets of $[n]$,*

$$\sum_{S \subseteq E} \Pr_B(1 \leq |S \cap F_B| \leq k) \widehat{f}(S)^2 \leq m^{-1/4} \frac{rm}{n} I_p(f). \quad (3.1)$$

Proof. Fix B and an assignment y to the coordinates in $E \setminus F_B$. Denote the resulting Boolean function on F_B by f_y . Every permutation supported on B fixes each edge outside F_B individually, and consequently fixes y . Thus f_y is invariant under all these permutations, regardless of the assignment.

Remark (notation). For each fixed B and outside assignment y , f_y is the restriction of f to $\{0, 1\}^{F_B}$; its Fourier coefficients and influences are computed on the free coordinates with bias p , and it is invariant under every permutation supported on B .

An edge with j vertices in B has an orbit of size $\binom{m}{j}$: its outside part is fixed and its inside part can be any j -subset of B . Since $1 \leq j \leq r < m$, this size is at least m . The free product measure is invariant under the same permutations, so influences are equal within each orbit. It follows that

$$I_e(f_y) \leq I(f_y)/m \quad (e \in F_B).$$

Lemma 2.1 therefore applies to every f_y .

Average its conclusion over the product law of y . For $A \subseteq F_B$, the Fourier expansion of f gives

$$\widehat{f_y}(A) = \sum_{D \subseteq E \setminus F_B} \widehat{f}(A \cup D) \chi_D(y).$$

Orthonormality in the assigned coordinates yields

$$\mathbb{E}_y [\widehat{f_y}(A)^2] = \sum_{D \subseteq E \setminus F_B} \widehat{f}(A \cup D)^2.$$

Also $\mathbb{E}_y I_e(f_y) = I_{e,p}(f)$ for every $e \in F_B$, by conditioning the disagreement event. Consequently

$$\sum_{S: 1 \leq |S \cap F_B| \leq k} \widehat{f}(S)^2 \leq m^{-1/4} \sum_{e \in F_B} I_{e,p}(f).$$

Finally average over B . For each fixed edge e , a union bound on its r vertices gives $\Pr_B(e \in F_B) \leq rm/n$, proving (3.1). $\square$

The next lemma extends the graph capture argument of [9, Lemma 3.3] to identify the hypergraph supports detected by this estimate. Here a support S is itself a set of distinct r -edges; its vertex union contains no isolated vertices.

Lemma 3.2 (Capturing a support at one vertex). *Let $n \geq r \geq 3$ and $1 \leq m \leq n$ be integers, let $k > 0$, and put*

$$\alpha = \frac{r}{r-1}, \quad L = (k/r)^\alpha.$$

Suppose

$$rL \frac{m-1}{n-1} \leq \frac{1}{2}. \tag{3.2}$$

For a uniform m -subset B of $[n]$ and every $S \subseteq E_{n,r}$ with $1 \leq |S| \leq L$,

$$\Pr_B(1 \leq |S \cap F_B| \leq k) \geq \frac{m}{2n}. \tag{3.3}$$

Proof. Fix such an S , write $s = |S|$, and let $V = \bigcup_{e \in S} e$ and $v = |V|$. Because $s \leq \binom{v}{r} \leq v^r$, the average degree of the hypergraph with edge set S is at most

$$\frac{rs}{v} \leq rs^{(r-1)/r} \leq k.$$

Choose a vertex $u \in V$ of degree $d_S(u)$ between 1 and k . This choice depends only on S , before B is sampled.

Remark (notation). $d_S(u) = |\{e \in S : u \in e\}|$ is the degree of the chosen vertex $u \in \bigcup_{e \in S} e$, with $1 \leq d_S(u) \leq \min\{s, k\}$; the choice of u depends only on S and is made before B is sampled.

Conditional on $u \in B$, each other vertex lies in B with probability $(m-1)/(n-1)$. The union bound and $v \leq rs \leq rL$ therefore give

$$\begin{aligned}\Pr_B(B \cap V = \{u\}) &\geq \frac{m}{n} \left(1 - (v-1) \frac{m-1}{n-1}\right) \\ &\geq \frac{m}{2n}.\end{aligned}$$

On this event, the edges of S meeting B are exactly those containing u . Thus $|S \cap F_B| = d_S(u) \in [1, k]$. $\square$

The captured set need not consist of a single edge: the block meets one support vertex, and retains every support edge incident with it. Also the vertex u may vary with S , since (3.3) is a separate probability bound for each nonnegative summand in (3.1).

Combining Lemmas 3.1 and 3.2 gives, whenever their hypotheses hold,

$$\sum_{1 \leq |S| \leq L} \hat{f}(S)^2 \leq 2rm^{-1/4} I_p(f). \quad (3.4)$$

If $L < 1$, this conclusion holds because the sum is empty. Otherwise, restrict the nonnegative sum in (3.1) to $1 \leq |S| \leq L$ and divide by the lower bound $m/(2n)$. This completes the transfer from restricted degree k to original degree $(k/r)^{r/(r-1)}$.

4 The uniform bound and its threshold consequence

We now choose the block size so that (3.4) controls the low-degree part of the variance. The weighted Parseval identity will control the remaining degrees.

Proof of Theorem 1.1. Fix $r \geq 3$ and write $\alpha = r/(r-1)$. Choose an integer $N_r > r$ such that, for every $n \geq N_r$ and $m = \lfloor \sqrt{n} \rfloor$,

$$\begin{aligned}m &\geq r+1, & \log m &\geq \tfrac{1}{3} \log n, \\ r(\log n)^\alpha n^{-1/2} &\leq \tfrac{1}{2}, & m^{-1/4}(\log n)^\alpha &\leq 1.\end{aligned} \quad (4.1)$$

Such an N_r exists: m is asymptotic to $\sqrt{n}$, and any fixed power of $\log n$ grows more slowly than a positive power of n . In particular, the choice is independent of p and f .

Remark (notation). $N_r > r$ is a fixed integer depending only on r , chosen so that all four inequalities in (4.1) hold for every $n \geq N_r$ with $m = \lfloor \sqrt{n} \rfloor$.

Assume first that $n \geq N_r$, and set

$$\sigma = \sqrt{p(1-p)}, \quad k = \frac{\sigma \log m}{64}, \quad L = (k/r)^\alpha.$$

Both cutoffs are positive. Since $k/r \leq \log n$ and

$$\frac{m-1}{n-1} \leq \frac{\sqrt{n}-1}{n-1} = \frac{1}{\sqrt{n}+1} \leq n^{-1/2},$$

the third condition in (4.1) implies (3.2). Thus (3.4) and the fourth condition in (4.1) give

$$\sum_{1 \leq |S| \leq L} \hat{f}(S)^2 \leq \frac{2r}{(\log n)^\alpha} I_p(f). \quad (4.2)$$

For the remaining degrees, (2.1) gives

$$\sum_{|S|>L} \hat{f}(S)^2 \leq \frac{1}{L} \sum_S |S| \hat{f}(S)^2 = \frac{\sigma^2}{L} I_p(f).$$

The decisive bias dependence is

$$\frac{\sigma^2}{L} = \frac{(64r)^\alpha \sigma^{2-\alpha}}{(\log m)^\alpha} \leq \frac{(192r)^\alpha}{(\log n)^\alpha}. \quad (4.3)$$

Indeed $2 - \alpha = (r - 2)/(r - 1) > 0$, $\sigma \leq 1$, and $\log m \geq (\log n)/3$. This estimate remains valid at arbitrarily small positive σ . If $L < 1$, the low-degree sum is empty and the same tail estimate bounds the whole variance.

Adding the two parts proves (1.1) for $n \geq N_r$ with constant $2r + (192r)^\alpha$. For $r \leq n < N_r$, (2.1) gives $\text{Var}_p(f) \leq \sigma^2 I_p(f) \leq I_p(f)/4$. Therefore a valid choice for all n and p is

$$C_r = \max \left\{ 2r + (192r)^\alpha, \frac{(\log N_r)^\alpha}{4} \right\}.$$

□

We finish with the threshold consequence stated in the introduction. Recall that $q(p) = \mathbb{E}_p f$ and $p_a = \inf\{p \in [0, 1] : q(p) \geq a\}$.

Corollary 4.1. *For a nontrivial increasing property of r -uniform hypergraphs on $n \geq r$ vertices, where $r \geq 3$, and every $0 < \varepsilon < 1/2$,*

$$p_{1-\varepsilon} - p_\varepsilon \leq \frac{2C_r}{(\log n)^{r/(r-1)}} \log \frac{1-\varepsilon}{\varepsilon}.$$

Proof. The Margulis–Russo identity relates pivotal influences to the probability derivative; see [8], [10, Lemma 3], and [5, Lemma 2.2]. Here it follows directly by giving each coordinate its own bias: the expectation is affine in each bias separately, and its partial derivative in coordinate e is $\mathbb{E}_p[f(x_{e \leftarrow 1}) - f(x_{e \leftarrow 0})]$. Monotonicity identifies this with $I_{e,p}(f)$. Differentiating along the common bias gives $q'(p) = I_p(f)$.

Nontrivial monotonicity gives $q(0) = 0$ and $q(1) = 1$. Some pair of configurations differing in one coordinate has different values, so its pivotal event has positive probability for every $0 < p < 1$. Hence $q'(p) > 0$ there, and the stated quantiles exist uniquely. Theorem 1.1 implies

$$\frac{q'(p)}{q(p)(1-q(p))} \geq \frac{(\log n)^{r/(r-1)}}{C_r}.$$

Integrating from p_ε to $p_{1-\varepsilon}$, using the antiderivative $\log(q/(1-q))$, proves the claim. □

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